

On the Relativistic Micro-Canonical Ensemble and Relativistic Kinetic Theory for N Relativistic Particles in Inertial and Non-Inertial Rest Frames.

David Alba

*Sezione INFN di Firenze
Polo Scientifico, via Sansone 1
50019 Sesto Fiorentino, Italy
E-mail alba@fi.infn.it*

Horace W. Crater

*The University of Tennessee Space Institute
Tullahoma, TN 37388 USA
E-mail: hcrater@utsi.edu*

Luca Lusanna

*Sezione INFN di Firenze
Polo Scientifico, Via Sansone 1
50019 Sesto Fiorentino (FI), Italy
E-mail: lusanna@fi.infn.it*

Abstract

A new formulation of relativistic classical mechanics allows a revisiting of old unsolved problems in relativistic kinetic theory and in relativistic statistical mechanics. In particular a definition of the relativistic micro-canonical partition function is given strictly in terms of the Poincaré generators of an interacting N-particle system both in the inertial and non-inertial rest frames. The non-relativistic limit allows a definition of both the inertial and non-inertial micro-canonical ensemble strictly in terms of the Galilei generators. Also the one-particle relativistic distribution function is defined and a new approach to the relativistic Boltzmann equation is delineated. Finally there are some comments on relativistic dissipative fluids.

February 22, 2012

I. INTRODUCTION

In Ref.[1] we developed a new version of relativistic classical and quantum mechanics (RCM and RQM) for systems of N positive-energy particles in the rest-frame instant form of dynamics previously defined in Refs. [2–6] (it uses 3+1 splittings of Minkowski space-time and radar 4-coordinates, as explained in Section II, to have a well posed Cauchy problem). The non-relativistic limit reproduces classical and quantum mechanics with a Hamilton-Jacobi description of the Newtonian center of mass.

In RCM there is a complete control of the relativistic collective variables, with the non-covariant canonical center of mass; covariant non-canonical Fokker-Pryce center of inertia; non-covariant non-canonical Møller center of energy, all expressed in terms of Poincaré generators, and replacing the Newtonian center of mass and collapsing on it for $c \rightarrow \infty$. As shown in Ref. [1], to avoid all the problems with the localization of the relativistic center of mass¹ and the instantaneous spreading of the center-of-mass wave packets in RQM (violation of causality) one has to describe it with frozen (non evolving) Jacobi data \vec{z}, \vec{h} ². As a consequence, an isolated system with total 4-momentum $P^\mu = Mc \left(\sqrt{1 + \vec{h}^2}; \vec{h} \right)$, where Mc is the invariant mass of the system, is described as an *external* decoupled (non-local, non-measurable) non-covariant frozen 3-center of mass \vec{z} ³ carrying a *pole-dipole* structure: the invariant mass Mc and the rest angular momentum (or spin) of the isolated system. As it will be shown in Section II, in the so-called Wigner 3-spaces (orthogonal to P^μ) of the rest frame, the N particles are described by Wigner spin-1 phase space 3-vectors $\vec{\eta}_i(\tau)$, $\vec{\kappa}_i(\tau)$ restricted by rest-frame conditions eliminating the *internal* center of mass. Therefore the N -particle system is described by the external center of mass with canonical variables \vec{z}, \vec{h} , and by $N-1$ relative canonical variables $\vec{\rho}_a(\tau), \vec{\pi}_a(\tau)$, $a = 1, \dots, N-1$. The invariant mass is the Hamiltonian. The particle world-lines $x_i^\mu(\tau)$ and the standard momenta $p_i^\mu(\tau)$ are derived quantities: they turn out to be *covariant but not canonical predictive coordinates and momenta*.

Given this classical framework it is possible to define a consistent RQM taking into account all the known information about relativistic bound states [4, 5, 7–9] and with a total control on the Poincaré generators (both external and internal, see Section II) also in presence of action-at-a-distance interactions.

Moreover, as shown in Ref.[3], it is possible to reformulate RCM in global non-inertial frames of Minkowski space-time by means of parametrized Minkowski theories of isolated systems (see Section II) and to define RQM in some of them [10]⁴. In this approach the transition from a non-inertial frame to another (either non-inertial or inertial) one is formulated as a gauge transformation. Therefore, at least at the classical level, one has gauge equivalence of the inertial and non-inertial dynamics. The open problem at the quantum level is the implementation of these gauge transformations as unitary gauge transformations

¹ Not being a covariant quantity it is described by a different pseudo-world-line in every inertial frame.

² The frozen nature of the variables \vec{z}, \vec{h} , implies the Hamilton-Jacobi description of the center of mass in the non-relativistic limit.

³ $\vec{z}/Mc = \vec{x}_{NW}(0)$ are the Cauchy data for the Newton-Wigner 3-position.

⁴ The same happens in the parametrized Galilei theories defined in Ref.[11]. This allows us to define non-relativistic quantum mechanics (QM) in Galilean non-inertial frames.

⁵. An important class of non-inertial frames are the non-inertial rest frames (see Section III): in them one can give the explicit form of all the quantities relevant in the inertial rest frame.

Also perfect fluids can be described with the 3+1 point of view, as shown in Refs. [13, 14], if one uses the action principle of Brown [15] (see Ref.[16] for a review of relativistic hydrodynamics).

Both particle systems [17] and perfect fluids [18] can be described in ADM tetrad gravity [19] by using 3+1 splittings of globally hyperbolic, asymptotically Minkowskian space-times without super-translations. For perfect fluids it can be shown that the standard energy-momentum tensor looks like the one of a dissipative fluid to every congruence of time-like observers with zero vorticity. The standard perfect fluid form holds only for the congruence of comoving observers associated with the unit 4-velocity of the fluid (the 1+3 point of view), which is in general non-surface-forming having non zero vorticity (so that it cannot be used for the Cauchy problem).

In this paper we want to use the framework of this RCM for N-particle systems, fully consistent with Lorentz signature and with the Poincaré group under control, to give a definition of the relativistic micro-canonical ensemble in relativistic statistical mechanics and of the one-particle distribution function in relativistic kinetic theory. We want to study the aspects of relativistic kinetic theory and relativistic statistical mechanics (see for instance Refs. [20, 21]) connected with inertial and non-inertial frames in Minkowski space-time without entering into the foundational problems of relativistic thermodynamics, but taking into account the problem of the relativistic center of mass and the implications of the Wigner covariance of our 3-vectors $\vec{\eta}_i(\tau)$, $\vec{\kappa}_i(\tau)$, for the transformation properties of the relevant distribution functions. Therefore we will not consider canonical and grand-canonical ensembles, which are not equivalent to the micro-canonical ensemble when long-range interactions are present, and we will define only the micro-canonical temperature.

All these topics are relevant for astrophysics, cosmology, Brownian motion, plasma physics, heavy ion collisions and quark-gluon plasma. A very rich bibliography on many of these arguments can be found in Ref.[22]. In Refs. [3, 16] there is also the inclusion of the electro-magnetic field in the radiation gauge, but the consequences of its presence in non-inertial frames for plasma physics (Vlasov equation) and magneto-hydrodynamics have still to be explored.

We will limit ourselves to the following arguments:

A) In Ref.[23] (see also its bibliography) it was said that the lack of understanding of RCM with action-at-a-distance potentials forced people to develop a relativistic kinetic theory of "world-lines" and not of particles. This was due to the fact that it was not known how to parametrize the world-lines of N relativistic particles in an arbitrary inertial frame in terms of a Lorentz scalar time-parameter so that their interactions depend only on space-like distances between pairs of particles and not on relative times (the basic problem in the theory

⁵ With particle systems this can be done only in rotating systems till now [10]. Instead the problem is completely open for the massive Klein-Gordon field due to the no-go theorem of Ref.[12] forbidding unitary evolution in non-inertial frames.

of relativistic bound states). In our new RCM all these problems are solved: it is clarified how to synchronize the clocks of the N particles and how to formulate a Cauchy problem⁶ in presence of action-at-a-distance, electro-magnetic and gravitational interactions. We will give the explicit form of the Poincaré generators for a simple model (not yet the most general one) of N positive-energy particles with action-at-a-distance potentials given in Section IV. Before we had only the 2-body model of Ref.[7] and the complicated N -particle model with Coulomb plus Darwin potentials of Refs.[4, 5, 8, 9] (see Appendix A; in Appendix B there is its general relativistic counterpart, without electro-magnetic field, deduced from Ref.[16]). This simple N -particle model allows us to find the Liouville operators for single particles in inertial and non-inertial frames. A crucial difference with respect to other approaches is the decoupling of the external (non-local, non-measurable) relativistic center of mass

Let us remark that in all the previous attempts to define relativistic kinetic theory and relativistic statistical mechanics starting from a system of N relativistic particles, like the ones in Refs.[24–27] (see also the review in Ref.[28]), there are either no interactions or ad hoc ansatzs to avoid the foundational problems of RCM (quoted in Refs.[1, 7, 29]), whose validity is out of control.

B) In Ref.[30] there is the evaluation of the standard extended (i.e. depending also on the conserved angular momentum and not only on the energy of the isolated system like the ordinary one) non-relativistic distribution function for the micro-canonical ensemble for a system of N particles interacting with the long range Newton gravitational interaction (see Ref.[31] for the status of long-range interactions in non-relativistic statistical mechanics and the non-equivalence of micro-canonical and canonical ensembles). This suggested that we look for a formulation of the (ordinary and extended) relativistic micro-canonical ensemble both in the inertial rest frame and in the non-inertial ones (where the inertial forces are long range forces) for N particle systems⁷. It turns out that the definitions of these distribution functions depend in a natural way on the ten internal Poincaré generators and that the ensemble depends only upon the canonical relative variables $\vec{\rho}_a, \vec{\pi}_a, a = 1, \dots, N - 1$, but not on the frozen Jacobi data of the external center of mass. The non-relativistic limit allows us to find the (ordinary and extended) Newtonian micro-canonical ensembles both in inertial and non-inertial rest frames of Galilei space-time by using the generators of the Galilei group in the Hamilton-Jacobi description of the center of mass. Now, unlike with the relativistic case, one can reintroduce the motion of the center of mass to recover the known definition of the distribution function. In the relativistic case the non-covariance of the Jacobi data \vec{z} makes the reintroduction of the external center of mass impossible. It is only possible to pass from an expression depending on the relative variables to one depending on the particle world-lines. We are able to evaluate explicitly our modified distribution function in the inertial non-relativistic case, but not in the inertial relativistic case (also in the standard approach its form is not known when $m \neq 0$ [33]). Let us remark that to our knowledge this is the first time that one has a definition of micro-canonical ensemble in relativistic and non-relativistic non-inertial frames. In relativistic non-inertial frames naive relative variables

⁶ Even if non physical we must give the Cauchy data on a space-like 3-space, to be able to use the existence and unicity theorem for the solution of partial differential equations to predict the future. The subsequent evolution is consistent with special relativity even in the case of action-at-a-distance interactions.

⁷ See Ref. [32] for the extended micro-canonical ensemble of the ideal relativistic *quantum* gas in relativistic inertial frames.

cannot be defined, but action-at-a-distance potentials $V(\vec{\rho}_a^2)$ can be described by using the Synge world-function like in general relativity [34].

In inertial frames the (ordinary or extended) distribution function is time independent consistent with the standard notion of *equilibrium*. This turns out to be true also in the non-inertial rest frames due to the fact that both the Galilei or Poincaré generators are asymptotic constants of motion at spatial infinity. Therefore *equilibrium* (at least in a passive viewpoint ⁸) can be defined also in non-inertial rest frames, notwithstanding the fact that the inertial forces are long-range independently from the type of inter-particle interactions. See also Ref.[36] (and its bibliography) and Ref.[37] for the problem of the dependence of the constitutive relations of continuum mechanics on the non-inertial frame in the non-relativistic framework.

C) By using the micro-canonical entropy it is possible to define the micro-canonical temperature $T_{(mc)}$ (see Refs. [38, 39]; see Refs.[40] for the case in which long range forces are present and $T_{(mc)}$ is the only reliable notion of temperature) and to show that it is a *Lorentz scalar*. Therefore, when the thermodynamical limit ($N, V \rightarrow \infty$ with $N/V = const.$, $T_{(mc)} \rightarrow T$) exists, also the canonical temperature T turns out to be a Lorentz scalar. This is our answer to the endless debate on the transformation properties of temperature under Lorentz boosts ($T = T_{rest}$, $T = T_{rest} (1 - v^2/c^2)^{1/2}$, $T = T_{rest} (1 - v^2/c^2)^{-1/2}$): see Ref.[41] for the formulation of the problem and Refs.[42] for recent contributions. In non-inertial frames $T_{(mc)}$ will be a functional of the inertial potentials.

D) Starting from the density function of a non-equilibrium relativistic Gibbs ensemble (assumed to transform like the relativistic micro-canonical distribution function to which it tends at equilibrium) we can give a definition of the one-particle distribution function $f(\vec{\eta}, \vec{\kappa}, \tau)$ with a statistical average. We only investigate the transformation properties of this one-particle distribution function under Poincaré transformations in the inertial relativistic rest frame. We can show that it is a *Lorentz scalar*: this is our answer to the long standing debate on the transformation properties of this distribution reviewed in Refs. [43–45].

In the inertial non-relativistic case one can recover the Maxwell-Boltzmann distribution function as the equilibrium solution of the non-relativistic Boltzmann equation, which can be derived (see for instance chapter 3 of Ref.[20]) as an approximation from the BBGKY (Bogoliubov - Born - Green - Kirkwood - Yvon) hierarchy for the coupled equations of motion of the s-particle distributions functions implied by the Liouville theorem using the Hamiltonian of the N-particle system. In the relativistic case in absence of a consistent RCM the Boltzmann equation is either postulated or derived from quantum field theory (see Refs.[46], [21], [47]) ⁹, and for free particles the equilibrium solution is the relativistic

⁸ This is a different problem from how to describe equilibrium in general relativity [35], where there are physical tidal degrees of freedom of the gravitational field and the equivalence principle forbids the existence of global inertial frames.

⁹ Starting from the existing relativistic kinetic theory [49, 50] one arrives at the *relativistic Boltzmann equation* of Ref.[47]. As shown in Ref.[23] if we consider a gas of charged massive particles interacting with external electro-magnetic and gravitational fields, the Hamilton equations of motion of a particle are $\frac{dx^\mu(\lambda)}{d\lambda} = p^\mu(\lambda)$, $\frac{dp^\mu(\lambda)}{d\lambda} = e F^\mu{}_\nu(x(\lambda)) p^\nu(\lambda) - \Gamma^\mu_{\alpha\beta}(x(\lambda)) p^\alpha(\lambda) p^\beta(\lambda) = F^\mu(x(\lambda), p(\lambda))$ (λ is an affine parameter; self-forces are not considered). Therefore, if $\hat{L} = p^\mu \frac{\partial}{\partial x^\mu} + F^\mu \frac{\partial}{\partial p^\mu}$ is the associated Liouville

Boltzmann-Jüttner distribution function [48]. By adapting this construction to the relativistic inertial rest frame, we show that the Jüttner distribution can be obtained also in our approach.

Since in our RCM we have the Hamilton equations for the interacting N-particle systems of Section IV (and of Appendices A and B) described by Wigner-covariant phase space 3-variables $\vec{\eta}_i(\tau)$, $\vec{\kappa}_i(\tau)$, $i = 1, \dots, N$, (or by the relative ones $\vec{\rho}_a(\tau)$, $\vec{\pi}_a(\tau)$, $a = 1, \dots, N - 1$, after the elimination of the internal center of mass in the Wigner 3-spaces), we can introduce also the correlation functions $f_s(\vec{\eta}_1, \vec{\kappa}_1, \dots, \vec{\eta}_s, \vec{\kappa}_s, \tau)$, $s = 1, \dots, N$, and, by using the Liouville operator associated to the relativistic N-particle Hamiltonian (the global invariant mass), find their coupled equations of motion. This produces a relativistic BBGKY hierarchy, from which the form of the relativistic Boltzmann equation implied by our RCM, with a decoupled external relativistic center of mass, could be defined in absence of external forces. However the problem is much more complex than in the non-relativistic case due to the form of the relativistic Hamiltonian and to the momentum-dependence of the potentials (see Section IV) and only for certain models and with drastic approximations can a Boltzmann equation be derived.

E) The moments of the one-particle distribution function, solution of the Boltzmann equation, give rise to a hydrodynamical description of relativistic kinetic theory with an effective dissipative fluid (to be contrasted with the perfect or dissipative fluids of relativistic hydrodynamics). This allows us to study the problems of relativistic dissipative fluids and of causal relativistic thermodynamics, whose foundations are not yet fully established ¹⁰, but which is used for instance as a hydrodynamical model for describing relativistic heavy-ion collisions [52]. See Refs.[53, 54] for a review of the Eckart, Landau-Lifschitz, Israel-Stewart [49] and Carter [55] points of view and Ref.[56] for the 1+3 point of view ¹¹.

The various approaches differ in the parametrization of the relativistic entropy current $S^\mu = s u^\mu + q^\mu + V^\mu$ in terms of the fluid 4-velocity u^μ , the ordinary entropy s , the heat transfer 4-vector q^μ and the viscous terms V^μ (with q^μ and V^μ orthogonal to u^μ), in the definition of u^μ and in the order of the deviations from equilibrium (linear for the Eckart and Landau-Lifschitz models, quadratic for the Israel-Stewart and Carter models). See Refs.[50] and the Appendix B of Ref.[13] for reviews; for recent developments see Refs.[57–59].

operator, the relativistic Boltzmann equation is $\hat{L}|_{\epsilon g^{\alpha\beta}(x) p_\alpha p_\beta = m^2 c^2} f(x^o, \vec{x}, \vec{p}) = \mathcal{C}[f]$, where $\mathcal{C}[f]$ is the collision term, bilinear in f for 2-body collisions. At equilibrium (no collisions) the distribution function $f_{eq}(x^o, \vec{x}, \vec{p})$ satisfies this equation with $\mathcal{C}[f_{eq}] = 0$ and $\partial_{x^o} f_{eq} = 0$.

¹⁰ For instance there are discussions going on whether the standard definition of relativistic thermodynamics given in Ref.[49]) is acceptable. See for instance Ref.[51] for a proposal to modify the first member of the Gibbs relation $de = Tds + \mu dn$ by adding a dependence on the momentum density to the variation de of the local rest frame energy density.

¹¹ In this paper we use a 3+1 point of view, which is defined in Subsection A of Section II. Instead in the 1+3 point of view one gives only the world-line of a time-like observer and the instantaneous 3-spaces are identified with the tangent spaces orthogonal to the unit 4-velocity of the observer in each point of the world-line. As shown in Ref.[3] these tangent planes intersect each other at a certain distance from the observer, so that coordinates like the Fermi or rotating ones can be defined only locally due to these coordinate singularities.

Then one has to implement the second law of thermodynamics in the form $\partial_\mu S^\mu \geq 0$. In Carter's two-fluid approach one writes $S^\mu = s u^\mu + j^\mu$ and considers j^μ as a second fluid; a Lagrangian-like approach is used to implement $\partial_\mu S^\mu \geq 0$. As shown in Ref.[60] Israel-Stewart and Carter approaches are essentially equivalent.

What is lacking in all these approaches is a variational principle describing relativistic fluids out of equilibrium and implying $\partial_\mu S^\mu \geq 0$. Our last contribution to the relativistic framework is the suggestion that the action principle for relativistic fluids of Ref.[15], in the version of Refs.[13, 14, 18], can be modified to get these results.

Since in Ref.[1] there is only the basic kinematics for the new RCM, in Section II we give a full review of it in the inertial rest frame (Subsections A and B). Then we give new results in Subsection C on how our description of N-particle systems appears in arbitrary inertial frames, since they are needed in the rest of the paper. In Subsection D we recall the non-relativistic limit.

Section III is devoted to the extension of the description of N-particle systems to admissible non-inertial frames (Subsection A) and to the non-inertial rest frames (Subsections B and C). In Subsection D the non-relativistic limit of non-inertial frames is considered.

Since in relativistic kinetic theory and in relativistic statistical mechanics we must consider N-particle systems, in Section IV we recall for which N-particle systems is known an explicit realization of the Poincaré generators and we add the construction of these generators for a new model, where they have the simplest form, to be used as an example in the subsequent Sections.

In Section V, after these preliminaries and after recalling the standard formulation of the micro-canonical ensemble in non-relativistic inertial frames (Subsection A), we give our new definition of it (and of its extension depending also on the angular momentum) in the non-relativistic inertial frames (Subsection B), in the relativistic inertial rest frames (Subsections C, D and E; in Subsection E there is the definition of the relativistic micro-canonical temperature), in relativistic non-inertial rest frames (Subsection F) and finally in non-relativistic non-inertial frames (Subsection G).

In Section VI we give our definition of the one-particle distribution function of relativistic statistical mechanics (Subsections A and B) and an introduction to which form of the relativistic Boltzmann equation is implied by our approach (Subsection C). In Subsection D we discuss the problem of how to define the one-particle distribution function in non-inertial frames.

In Section VII there is a proposal for a modification of an existing action principle for relativistic fluids so to reproduce some results on relativistic dissipative fluids.

Section VIII contains the Conclusions and a discussion of the open problems.

In Appendix A there is the form of the Poincaré generators for N charged scalar particles with Coulomb plus Darwin mutual interaction.

In Appendix B there is the form of the ADM Poincaré generators for N scalar particles in the framework of Hamiltonian Post-Minkowskian tetrad gravity.

In Appendix C there the explicit calculations for the non-relativistic micro-canonical partition function for N free particles.

II. CLASSICAL RELATIVISTIC N-BODY SYSTEMS IN THE INERTIAL REST FRAME

Let us consider an isolated system of N positive-energy scalar particles either free or interacting.

A. Classical Relativistic Mechanics in the Rest-Frame Instant Form

Let give a review of relativistic mechanics of isolated systems in special relativity following the approach of Refs.[1–7].

As shown in Ref.[3] we now have a metrology-oriented description of non-inertial frames in special relativity. This can be done with the *3+1 point of view* and the use of observer-dependent Lorentz-scalar radar 4-coordinates. Let us give the world-line $x^\mu(\tau)$ of an arbitrary time-like observer carrying a standard atomic clock: τ is an arbitrary monotonically increasing function of the proper time of this clock. Then we give an admissible 3+1 splitting of Minkowski space-time, namely a nice foliation with space-like instantaneous 3-spaces Σ_τ : it is the mathematical idealization of a protocol for clock synchronization (all the clocks in the points of Σ_τ sign the same time of the atomic clock of the observer). On each 3-space Σ_τ we choose curvilinear 3-coordinates σ^r having the observer as origin. These are the *radar 4-coordinates* $\sigma^A = (\tau; \sigma^r)$. If $x^\mu \mapsto \sigma^A(x)$ is the coordinate transformation from the Cartesian 4-coordinates x^μ of a reference inertial observer to radar coordinates, its inverse $\sigma^A \mapsto x^\mu = z^\mu(\tau, \sigma^r)$ defines the *embedding* functions $z^\mu(\tau, \sigma^r)$ describing the 3-spaces Σ_τ as an embedded 3-manifold into Minkowski space-time. From now on we shall denote the curvilinear 3-coordinates σ^r with the notation $\vec{\sigma}$ for the sake of simplicity.

The induced 4-metric on Σ_τ is the following functional of the embedding: ${}^4g_{AB}(\tau, \vec{\sigma}) = [z_A^\mu \eta_{\mu\nu} z_B^\nu](\tau, \vec{\sigma})$, where $z_A^\mu = \partial z^\mu / \partial \sigma^A$ and $\eta_{\mu\nu}$ is the flat metric. The 4-metric ${}^4g_{AB}$ has signature $\epsilon(+---)$ with $\epsilon = \pm$ (the particle physics, $\epsilon = +$, and general relativity, $\epsilon = -$, conventions); the flat Minkowski metric is $\eta_{\mu\nu} = \epsilon(+---)$.

While the 4-vectors $z_r^\mu(\tau, \vec{\sigma})$ are tangent to Σ_τ , so that the unit normal $l^\mu(\tau, \vec{\sigma})$ is proportional to $\epsilon^\mu_{\alpha\beta\gamma} [z_1^\alpha z_2^\beta z_3^\gamma](\tau, \vec{\sigma})$, we have $z_r^\mu(\tau, \vec{\sigma}) = [N l^\mu + n^r z_r^\mu](\tau, \vec{\sigma})$ for the so-called evolution 4-vector, where $N(\tau, \vec{\sigma}) = 1 + n(\tau, \vec{\sigma}) = \epsilon [z_r^\mu l_\mu](\tau, \vec{\sigma})$ and $n_r(\tau, \vec{\sigma}) = -\epsilon g_{rr}(\tau, \vec{\sigma}) = [{}^3g_{rs} n^s](\tau, \vec{\sigma})$ are the lapse and shift functions. We also have $|det {}^4g| = (1 + n) \sqrt{\gamma}$; $\sqrt{\gamma} = \sqrt{det {}^3g}$ with ${}^3g_{rs} = -\epsilon {}^4g_{rs}$ of positive signature.

The conditions for having an admissible 3+1 splitting of space-time are:

- a) $1 + n(\tau, \vec{\sigma}) > 0$ everywhere (the instantaneous 3-spaces never intersect each other, so that there are no coordinate singularities as happens with Fermi coordinates);
- b) the Møller conditions, which imply
 - i) $\epsilon {}^4g_{\tau\tau} > 0$, i.e. $(1 + n)^2 > \sum_r n_r n^r$ (the rotational velocity never exceeds the velocity of light c , so that there no coordinate singularities as happens with the rotating disk);
 - ii) $\epsilon {}^4g_{rr} = -{}^3g_{rr} < 0$ (satisfied by the signature of ${}^3g_{rs}$), ${}^4g_{rr} {}^4g_{ss} - ({}^4g_{rs})^2 > 0$ and $det \epsilon {}^4g_{rs} = -det {}^3g_{rs} < 0$ (satisfied by the signature of ${}^3g_{rs}$) so that $det {}^4g_{AB} < 0$ (these conditions imply that ${}^3g_{rs}$ has three definite positive eigenvalues $\lambda_r = \Lambda_r^2$ in the non-degenerate case without Killing symmetries, the only one we consider);
- c) the 3-spaces Σ_τ tend in a direction-independent way to space-like hyper-planes (all parallel) at spatial infinity.

In this 3+1 point of view the embedding functions $z^\mu(\tau, \vec{\sigma})$ describe the inertial effects present in the given non-inertial frame. The 4-metric ${}^4g_{AB}(\tau, \vec{\sigma})$ is the potential for the induced inertial effects. For instance the extrinsic curvature ${}^3K_{rs}(\tau, \vec{\sigma}) = \left(\frac{1}{2(1+n)} (n_{r|s} + n_{s|r} - \partial_\tau {}^3g_{rs}) \right)(\tau, \vec{\sigma})$ of the non-Euclidean 3-spaces Σ_τ ($|r$ denotes the covariant derivative in it) is one of these induced inertial effects. All these inertial effects derive from the gauge freedoms of clock synchronization and choice of the 3-coordinates.

In Ref.[3] there is a complete description of the isolated systems admitting a Lagrangian description in non-inertial frames by means of *parametrized Minkowski theories*. In them there is a well defined action principle containing the embeddings $z^\mu(\tau, \vec{\sigma})$ as Lagrangian variables and allowing the determination of the energy-momentum tensor $T^{\mu\nu}(z(\tau, \vec{\sigma})) = \left(z_A^\mu z_B^\nu T^{AB} \right)(\tau, \vec{\sigma})$ of the isolated system. This allows us to find the ten Poincaré generators and to study the configurations of the isolated system having them finite. We shall only consider the case in which the total 4-momentum is time-like.

The embeddings turn out to be gauge variables (i.e. the transition from a frame to another one is a gauge transformation), because their conjugate canonical momenta $\rho_\mu(\tau, \vec{\sigma})$ are determined by four first class constraints (a de-parametrization of the super-momentum and super-Hamiltonian constraints of general relativity; they exist due to the invariance of the action principle under frame-preserving diffeomorphisms)

$$\begin{aligned} \rho_\mu(\tau, \vec{\sigma}) &= \left(\sqrt{-g} z_{A\mu} T^{\tau A} \right)(\tau, \vec{\sigma}) = \\ &= \left((1+n)^2 \sqrt{\gamma} T^{\tau\tau} l_\mu + (1+n) \sqrt{\gamma} \left[T^{\tau r} + T^{\tau\tau} n^r \right] z_{r\mu} \right)(\tau, \vec{\sigma}) = \\ &\stackrel{def}{=} \left(\sqrt{\gamma} \left[l_\mu T_{\perp\perp} - z_{r\mu} h^{rs} T_{\perp s} \right] \right)(\tau, \vec{\sigma}). \end{aligned} \quad (2.1)$$

The ten Poincaré generators are

$$P^\mu = \int d^3\sigma \rho^\mu(\tau, \vec{\sigma}), \quad J^{\mu\nu} = \int d^3\sigma \left(z^\mu \rho^\nu - z^\nu \rho^\mu \right)(\tau, \vec{\sigma}). \quad (2.2)$$

In special relativity we can restrict ourselves to inertial frames and define the *inertial rest-frame instant form of dynamics for isolated systems* by choosing the 3+1 splitting corresponding to the intrinsic inertial rest frame of the isolated system centered on an inertial observer: the instantaneous 3-spaces, named Wigner 3-space due to the fact that the 3-vectors inside them are Wigner spin-1 3-vectors [3], are orthogonal to the conserved 4-momentum P^μ of the configuration.

In Ref.[3] there is the extension to admissible *non-inertial rest frames*, where P^μ is orthogonal to the asymptotic space-like hyper-planes to which the instantaneous 3-spaces tend at spatial infinity.

The simplest form of the embedding of the Wigner 3-spaces in Minkowski space-time described in the inertial frame of an arbitrary inertial observer is

$$z_W^\mu(\tau, \vec{\sigma}) = Y^\mu(\tau) + \epsilon_r^\mu(\vec{h}) \sigma^r = Y^\mu(0) + \Lambda^\mu_A(\vec{h}) \sigma^A, \quad (2.3)$$

where $Y^\mu(\tau) = Y^\mu(0) + h^\mu \tau = z_W^\mu(\tau, \vec{0})$ is the world-line of the external Fokker-Pryce 4-center of inertia with $\eta_{\mu\nu} \epsilon_A^\mu(\vec{h}) \epsilon_B^\nu(\vec{h}) = \eta_{AB}$ and

$$\begin{aligned} \epsilon_\tau^\mu(\vec{h}) &= \frac{P^\mu}{Mc} = u^\mu(P) = h^\mu = \left(\sqrt{1 + \vec{h}^2}; \vec{h} \right) = \Lambda^\mu_\tau(\vec{h}), \quad \epsilon P^2 = M^2 c^2, \\ \epsilon_r^\mu(\vec{h}) &= \left(h_r; \delta_r^i + \frac{h^i h_r}{1 + \sqrt{1 + \vec{h}^2}} \right) = \Lambda^\mu_r(\vec{h}). \end{aligned} \quad (2.4)$$

Since the 3-metric inside the Wigner 3-space is Euclidean with positive signature we have $h^i = h_i$. The Lorentz matrix $\Lambda^\mu_A(\vec{h})$ is obtained from the standard Wigner boost $\Lambda^\mu_\nu(P^\alpha/Mc)$, sending the time-like 4-vector P^μ/Mc into $(1; 0)$, by transforming the index ν into an index adapted to radar 4-coordinates ($\Lambda^\mu_\nu \mapsto \Lambda^\mu_A$).

The form of this embedding is a consequence of the clarification of the notion of relativistic center of mass of an isolated system after a century of research. It turns out that there are only three notions of collective variables, which can be built using *only* the Poincaré generators (they are *non-local* quantities expressed as integrals over the whole Σ_τ) of the isolated system: the canonical non-covariant Newton-Wigner 4-center of mass (or center of spin) $\tilde{x}^\mu(\tau)$, the non-canonical covariant Fokker-Pryce 4-center of inertia $Y^\mu(\tau)$ and the non-canonical non-covariant Møller 4-center of energy $R^\mu(\tau)$. All of them tend to the Newtonian center of mass in the non-relativistic limit.

These three variables can be expressed as known functions:

- a) of the Lorentz-scalar rest time $\tau = c T_s = h \cdot \tilde{x} = h \cdot Y = h \cdot R$;
- b) of canonically conjugate Jacobi data (frozen Cauchy data) $\vec{h} = \vec{P}/Mc$ and $\vec{z} = Mc \vec{x}_{NW}(0)$ ($\{z^i, h^j\} = \delta^{ij}$)¹²;
- c) of the invariant mass $Mc = \sqrt{\epsilon P^2}$ of the isolated system;
- d) of the rest spin \vec{S} of the isolated system.

The three collective variables have the following expression.

- 1) The pseudo-world-line of the canonical non-covariant external 4-center of mass is

$$\begin{aligned} \tilde{x}^\mu(\tau) &= \left(\tilde{x}^0(\tau); \tilde{\vec{x}}(\tau) \right) = \left(\sqrt{1 + \vec{h}^2} \left(\tau + \frac{\vec{h} \cdot \vec{z}}{Mc} \right); \frac{\vec{z}}{Mc} + \left(\tau + \frac{\vec{h} \cdot \vec{z}}{Mc} \right) \vec{h} \right) = \\ &= z_W^\mu(\tau, \vec{\sigma}) = Y^\mu(\tau) + \left(0; \frac{-\vec{S} \times \vec{h}}{Mc(1 + \sqrt{1 + \vec{h}^2})} \right), \end{aligned} \quad (2.5)$$

¹² The 3-vector $\vec{x}_{NW}(\tau)$ is the standard Newton-Wigner non-covariant 3-position, classical counterpart of the corresponding position operator; the use of \vec{z} avoids taking into account the mass spectrum of the isolated system in the description of the center of mass.

with $\tilde{\sigma} = \frac{-\vec{S} \times \vec{h}}{Mc(1+\sqrt{1+\vec{h}^2})}$ giving its location on the Wigner 3-space.

2) The world-line of the non-canonical covariant external Fokker-Pryce 4-center of inertia (origin of the Wigner 3-space) is

$$\begin{aligned} Y^\mu(\tau) &= \left(\tilde{x}^o(\tau); \vec{Y}(\tau) \right) = \left(\sqrt{1+\vec{h}^2} \left(\tau + \frac{\vec{h} \cdot \vec{z}}{Mc} \right); \frac{\vec{z}}{Mc} + \left(\tau + \frac{\vec{h} \cdot \vec{z}}{Mc} \right) \vec{h} + \frac{\vec{S} \times \vec{h}}{Mc(1+\sqrt{1+\vec{h}^2})} \right) = \\ &= z_W^\mu(\tau, \vec{0}), \end{aligned} \quad (2.6)$$

$$\text{with } Y^\mu(0) = \left(\sqrt{1+\vec{h}^2} \frac{\vec{h} \cdot \vec{z}}{Mc}; \frac{\vec{z}}{Mc} + \frac{\vec{h} \cdot \vec{z}}{Mc} \vec{h} + \frac{\vec{S} \times \vec{h}}{Mc(1+\sqrt{1+\vec{h}^2})} \right).$$

3) The pseudo-world-line of the non-canonical non-covariant external Møller 4-center of energy is

$$\begin{aligned} R^\mu(\tau) &= \left(\tilde{x}^o(\tau); \vec{R}(\tau) \right) = \left(\sqrt{1+\vec{h}^2} \left(\tau + \frac{\vec{h} \cdot \vec{z}}{Mc} \right); \right. \\ &\quad \left. \frac{\vec{z}}{Mc} + \left(\tau + \frac{\vec{h} \cdot \vec{z}}{Mc} \right) \vec{h} - \frac{\vec{S} \times \vec{h}}{Mc\sqrt{1+\vec{h}^2}(1+\sqrt{1+\vec{h}^2})} \right) = \\ &= z_W^\mu(\tau, \vec{\sigma}_R) = Y^\mu(\tau) + \left(0; \frac{-\vec{S} \times \vec{h}}{Mc\sqrt{1+\vec{h}^2}} \right), \end{aligned} \quad (2.7)$$

with $\vec{\sigma}_R = \frac{-\vec{S} \times \vec{h}}{Mc\sqrt{1+\vec{h}^2}}$ giving its location on the Wigner 3-space.

While $Y^\mu(\tau)$ is a 4-vector, $\tilde{x}^\mu(\tau)$ and $R^\mu(\tau)$ are not 4-vectors.

The transformation properties under Poincaré transformations (a, Λ) of \vec{h} , \vec{z} , τ , $\tilde{x}^\mu(\tau)$, $R^\mu(\tau)$ are given in Ref.[1].

Every isolated system (i.e. a closed universe) can be visualized as a decoupled non-covariant collective (non-local) pseudo-particle described by the frozen Jacobi data \vec{z} , \vec{h} carrying a *pole-dipole structure*, namely the invariant mass Mc and the rest spin \vec{S} of the system, and with an associated *external* realization of the Poincaré group (the last term in the Lorentz boosts induces the Wigner rotation of the 3-vectors inside the Wigner 3-spaces):

$$\begin{aligned} P^\mu &= Mc h^\mu = Mc \left(\sqrt{1+\vec{h}^2}; \vec{h} \right), \\ J^{ij} &= z^i h^j - z^j h^i + \epsilon^{ijk} S^k, \quad K^i = J^{oi} = -\sqrt{1+\vec{h}^2} z^i + \frac{(\vec{S} \times \vec{h})^i}{1+\sqrt{1+\vec{h}^2}}, \end{aligned} \quad (2.8)$$

satisfying the Poincaré algebra: $\{P^\mu, P^\nu\} = 0$, $\{P^\mu, J^{\alpha\beta}\} = \eta^{\mu\alpha} P^\beta - \eta^{\mu\beta} P^\alpha$, $\{J^{\mu\nu}, J^{\alpha\beta}\} = C_{\gamma\delta}^{\mu\nu\alpha\beta} J^{\gamma\delta}$, $C_{\gamma\delta}^{\mu\nu\alpha\beta} = \delta_\gamma^\nu \delta_\delta^\alpha \eta^{\mu\beta} + \delta_\gamma^\mu \delta_\delta^\beta \eta^{\nu\alpha} - \delta_\gamma^\nu \delta_\delta^\beta \eta^{\mu\alpha} - \delta_\gamma^\mu \delta_\delta^\alpha \eta^{\nu\beta}$.

The Jacobi data \vec{z} can be written in the form $\vec{z} = Mc \vec{R} + \frac{Mc \vec{S} \times \vec{P}}{P^o (Mc + P^o)}$, with $\vec{S} = \vec{J} - \vec{z} \times \frac{\vec{P}}{Mc}$, and this implies $\vec{z} = -\frac{P^o}{Mc} \vec{K} + \frac{\vec{P} \times \vec{K}}{P^o (Mc + P^o)} \vec{P} + \frac{\vec{J} \times \vec{P}}{Mc + P^o}$. Eq.(2.8) then allows us to express the external 4-center of mass $\tilde{x}^\mu(\tau)$ in terms of the external Poincaré generators [6]. The same can be done for $Y^\mu(\tau)$ by using Eq.(2.6). Therefore the three collective variables of an isolated relativistic system are *non-local* quantities like the Poincaré generators.

The universal breaking of Lorentz covariance is connected to this decoupled non-local collective variable and is irrelevant because all the dynamics of the isolated system lives inside the Wigner 3-spaces and is Wigner-covariant. Inside these Wigner 3-spaces the system is described by an internal 3-center of mass with a conjugate 3-momentum¹³ and by relative variables and there is an *unfaithful internal* realization of the Poincaré group (whose generators are determined by using the energy-momentum tensor of the isolated system): the internal 3-momentum, conjugate to the internal 3-center of mass, vanishes due the rest-frame condition. To avoid a double counting of the center of mass, i.e. an external one and an internal one, the internal (interaction-dependent) Lorentz boosts must also vanish. The only non-zero internal generators are the invariant mass Mc and the rest spin \vec{S} and the dynamics is re-expressed only in terms of internal Wigner-covariant relative variables.

The generators of the unfaithful internal realization of the Poincaré' algebra determined by the energy-momentum tensor (in inertial frames Eqs.(2.1) imply $T_{\perp\perp} = T^{\tau\tau}$ and $T_{\perp r} = \delta_{rs} T^{\tau s}$) are

$$\begin{aligned} Mc &= \int d^3\sigma T^{\tau\tau}(\tau, \vec{\sigma}), & \bar{S}^r &= \frac{1}{2} \delta^{rs} \epsilon_{su\nu} \int d^3\sigma \sigma^u T^{\tau\nu}(\tau, \vec{\sigma}), \\ \mathcal{P}^r &= \int d^3\sigma T^{\tau r}(\tau, \vec{\sigma}) \approx 0, & \mathcal{K}^r &= - \int d^3\sigma \sigma^r T^{\tau\tau}(\tau, \vec{\sigma}) \approx 0. \end{aligned} \quad (2.9)$$

The constraints $\vec{\mathcal{P}} \approx 0$ are the rest-frame conditions identifying the inertial rest frame. Having chosen the Fokker-Pryce center of inertia as origin of the 3-coordinates, the (*interaction-dependent*) constraints $\vec{\mathcal{K}} \approx 0$ are their gauge fixing: they eliminate the internal 3-center of mass so not to have a double counting (external, internal). Therefore the isolated system is described by the external non-covariant 3-center of mass \vec{z} , \vec{h} , and by an *internal space* of Wigner-covariant relative variables (M and \vec{S} depend only upon them).

As shown in Ref.[3] Eqs. (2.8) and (2.9) are obtained by putting the embedding (2.3) into Eq.(2.1): this implies $P^\mu = Mc h^\mu$ if $Mc = \int d^3\sigma T^{\tau\tau}(\tau, \vec{\sigma})$ and $\mathcal{P}^r = \int d^3\sigma T^{\tau r}(\tau, \vec{\sigma}) \approx 0$. Then Eq.(2.2) together with Eq.(2.6) for $Y^\mu(\tau)$ determine the form of J^i , K^i , \bar{S}^r , $\mathcal{K}^r \approx 0$.

As shown in Ref.[3], the restriction of the embedding $z^\mu(\tau, \sigma^u)$ to the Wigner 3-spaces implies the replacement of the Dirac Hamiltonian with the new one

¹³ Due to the rest-frame condition $\vec{\mathcal{P}} \approx 0$ of Eq.(2.9), we have $\vec{q}_+ \approx \vec{R}_+ \approx \vec{y}_+$, where \vec{q}_+ is the internal canonical 3-center of mass (the internal Newton-Wigner position), \vec{y}_+ is the internal Fokker-Pryce 3-center of inertia and \vec{R}_+ is the internal Møller 3-center of energy. As a consequence there is a unique internal 3-center of mass, which is eliminated by the vanishing of the internal Lorentz boosts.

$$H = Mc. \quad (2.10)$$

Therefore, the effective Hamiltonian is the invariant mass of the isolated system, whose conserved rest spin is \vec{S} .

The form (2.3) of the embedding of the Wigner 3-spaces into Minkowski space-time holds when the internal center of mass is eliminated with the rest-frame conditions (elimination of the internal 3-momentum and internal Lorentz boost) $\vec{\mathcal{P}} \approx 0$ and $\vec{\mathcal{K}} \approx 0$ [3]¹⁴.

In each Lorentz frame one has different pseudo-world-lines describing R^μ and \tilde{x}^μ : the canonical 4-center of mass \tilde{x}^μ lies in between Y^μ and R^μ in every (non rest)-frame. This leads to the existence of the *Møller non-covariance world-tube*, around the world-line Y^μ of the covariant non-canonical Fokker-Pryce 4-center of inertia Y^μ . The *invariant radius* of the tube is $\rho = \sqrt{-W^2/p^2} = |\vec{S}|/\sqrt{P^2}$ where ($W^2 = -P^2 \vec{S}^2$ is the Pauli-Lubanski invariant when $P^2 > 0$). This classical intrinsic radius delimits the non-covariance effects (the pseudo-world-lines) of the canonical 4-center of mass \tilde{x}^μ ¹⁵. They are not detectable because the Møller radius is of the order of the Compton wave-length: an attempt to test its interior would mean to enter in the quantum regime of pair production.

For N free positive energy spinless particles their world-lines are parametrized in terms of Wigner 3-vectors $\vec{\eta}_i(\tau)$, $i = 1, \dots, N$, in the following way

$$x_i^\mu(\tau) = z_W^\mu(\tau, \vec{\eta}_i(\tau)) = Y^\mu(\tau) + \epsilon_r^\mu(\vec{h}) \eta_i^r(\tau) = Y^\mu(0) + \Lambda^\mu_A(\vec{h}) \eta_i^A(\tau),$$

$$\eta_i^A(\tau) = (\tau; \eta_i^r(\tau)). \quad (2.11)$$

At the Hamiltonian level the basic canonical variables describing the particle are $\vec{\eta}_i(\tau)$ and their canonically conjugate momenta $\vec{\kappa}_i(\tau)$: $\{\eta_i^r(\tau), \kappa_j^s(\tau)\} = \delta_{ij} \delta^{rs}$. The standard momenta of the positive-energy scalar particles are

$$p_i^\mu(\tau) = \Lambda^\mu_A(\vec{h}) \kappa_i^A(\tau), \quad \kappa_i^A(\tau) = (E_i(\tau); \kappa_{ir}(\tau)). \quad (2.12)$$

For free particles we have $E_i(\tau) = \sqrt{m_i^2 c^2 + \vec{\kappa}_i^2(\tau)}$, $\epsilon p_i^2 = m_i^2 c^2$ and $Mc = \sum_i E_i$.

In the interacting case it is $E_i \neq \sqrt{m_i^2 c^2 + \vec{\kappa}_i^2(\tau)}$ and $\epsilon p_i^2 \neq m_i^2 c^2$. Instead $E_i(\tau)$ must be deduced from the form of the invariant mass Mc , which is the Hamiltonian for the τ -evolution in the Wigner 3-spaces. In the two-body model of Ref.[7] and in the N-body model of Section IV, the invariant mass has the form¹⁶ $Mc = \sum_i E_i(\tau)$ with $E_i(\tau) = \sqrt{m_i^2 c^2 + \vec{\kappa}_i^2(\tau) + V_i(\tau)}$,

¹⁴ If one modifies $\vec{\mathcal{K}} \approx 0$ to $\vec{\mathcal{K}} \approx F(\vec{\tau})$ the Fokker-Pryce center of inertia $Y^\mu(\tau)$ (an inertial observer; being non-locally defined is not measurable) is replaced by another 4-vector describing a non-inertial time-like observer $x^\mu(\tau)$ as origin of the radar 3-coordinate on the Wigner 3-space.

¹⁵ In the rest-frame the world-tube is a cylinder: in each instantaneous 3-space there is a disk of possible positions of the canonical 3-center of mass orthogonal to the spin. In the non-relativistic limit the radius ρ of the disk tends to zero and we recover the non-relativistic center of mass.

¹⁶ Instead the rest spin is always $\vec{S} = \sum_i \vec{\eta}_i \times \vec{\kappa}_i$ being in an instant form of dynamics.

where $V_i(\tau) = \tilde{V}_i(\vec{\eta}_m(\tau) - \vec{\eta}_n(\tau), \vec{\kappa}_k)$ are suitable action-at-a-distance potentials. This implies $\epsilon p_i^2 = m_i^2 c^2 + V_i(\tau)$. In these cases we have $\dot{\vec{\eta}}_i(\tau) = \{\vec{\eta}_i(\tau), Mc\} = \frac{\vec{\kappa}_i(\tau)}{E_i(\tau)}$ and $p_i^\mu(\tau) = E_i(\tau) \dot{x}_i^\mu(\tau)$ with $\dot{x}_i^\mu(\tau) = h^\mu + \epsilon_r^\mu(\vec{h}) \eta_i^r(\tau)$ from Eqs. (2.11) and (2.6) (so that we have $\epsilon \dot{x}_i^2(\tau) = \frac{m_i^2 c^2 + V_i(\tau)}{E_i^2(\tau)}$).

This description is not in contrast with scattering theory, where $\epsilon p_i^2 = m_i^2 c^2$ holds asymptotically for the *in* and *out* free particles (no interpolating description due to Haag no-go theorem for the interaction picture: see Ref.[4] for the way out, at least at the classical level, from this theorem in the 3+1 point of view), if the action-at-a-distance potentials (and also interactions with electro-magnetic fields) go to zero for large separations of the particles inside Wigner 3-spaces (the *cluster separability* in action-at-a-distance theories). In relativistic kinetic theory and in relativistic statistical mechanics $E_i = \sqrt{m_i^2 c^2 + \vec{\kappa}_i^2(\tau)}$ holds for a gas of non-interacting particles, otherwise it has to be replaced with an expression dictated by the type of the existing interactions.

For N free positive energy spinless particles the energy-momentum tensor is [3] ($z_\tau^\mu = (1+n)l^\mu + n^r z_r^\mu$; this expression holds in non-inertial frames; in Ref.[3] also an electro-magnetic field is present)

$$\begin{aligned}
T^{\mu\nu} &= z_A^\mu z_B^\nu T^{AB} = l^\mu l^\nu T_{\perp\perp} + (l^\mu z_r^\nu + l^\nu z_r^\mu) h^{rs} T_{\perp s} + z_r^\mu z_s^\nu T^{rs}, \\
T_{\perp\perp} &= l_\mu l_\nu T^{\mu\nu} = (1+n)^2 T^{\tau\tau}, \\
T_{\perp r} &= l_\mu z_{r\nu} T^{\mu\nu} = -(1+n) h_{rs} (T^{\tau\tau} n^s + T^{\tau s}), \\
T_{rs} &= z_{r\mu} z_{s\nu} T^{\mu\nu} = n_r n_s T^{\tau\tau} + (n_r h_{su} + n_s h_{ru}) T^{\tau u} + h_{ru} h_{sv} T^{uv}, \\
T_{\perp\perp}(\tau, \vec{\sigma}) &= \sum_{i=1}^N \frac{\delta^3(\sigma^u - \eta_i^u(\tau))}{\sqrt{\gamma(\tau, \vec{\sigma})}} \sqrt{m_i^2 c^2 + h^{rs} \kappa_{ir}(\tau) \kappa_{is}(\tau)}, \\
T_{\perp s}(\tau, \vec{\sigma}) &= - \sum_{i=1}^N \frac{\delta^3(\sigma^u - \eta_i^u(\tau))}{\sqrt{\gamma(\tau, \vec{\sigma})}} \kappa_{is}, \\
T_{rs}(\tau, \vec{\sigma}) &= \sum_{i=1}^N \frac{\delta^3(\sigma^u - \eta_i^u(\tau))}{\sqrt{\gamma(\tau, \vec{\sigma})}} \frac{\kappa_{ir} \kappa_{is}}{\sqrt{m_i^2 c^2 + h^{uv}(\tau, \vec{\sigma}) \kappa_{iu}(\tau) \kappa_{iv}(\tau)}},
\end{aligned} \tag{2.13}$$

and the internal generators have the following expression in the rest frame

$$\begin{aligned}
M c &= \frac{1}{c} \mathcal{E}_{(int)} = \sum_{i=1}^N \sqrt{m_i^2 c^2 + \vec{\kappa}_i^2}, \\
\vec{\mathcal{P}} &= \sum_{i=1}^2 \vec{\kappa}_i \approx 0, \\
\vec{S} &= \vec{\mathcal{J}} = \sum_{i=1}^2 \vec{\eta}_i \times \vec{\kappa}_i, \\
\vec{\mathcal{K}} &= - \sum_{i=1}^2 \vec{\eta}_i \sqrt{m_i^2 c^2 + \vec{\kappa}_i^2} \approx 0.
\end{aligned} \tag{2.14}$$

In the two-body case (see Refs.[3–5, 7] for the N-body case), by introducing the notation $\vec{\eta}_+$, $\vec{\kappa}_+ = \vec{\mathcal{P}}$, with a canonical transformation we get the following collective and relative variables

$$\begin{aligned}
\vec{\eta}_+ &= \frac{m_1}{m} \vec{\eta}_1 + \frac{m_2}{m} \vec{\eta}_2, & \vec{\rho} &= \vec{\eta}_1 - \vec{\eta}_2, \\
\vec{\kappa}_+ &= \vec{\kappa}_1 + \vec{\kappa}_2 \approx 0, & \vec{\pi} &= \frac{m_2}{m} \vec{\kappa}_1 - \frac{m_1}{m} \vec{\kappa}_2, \\
\vec{\eta}_i &= \vec{\eta}_+ + (-)^{i+1} \frac{m_{i+1}}{m} \vec{\rho}, & \vec{\kappa}_i &= \frac{m_i}{m} \vec{\kappa}_+ + (-)^{i+1} \vec{\pi},
\end{aligned} \tag{2.15}$$

where we use the convention $m_3 \equiv m_1$.

The collective variable $\vec{\eta}_+(\tau)$ has to be determined in terms of $\vec{\rho}(\tau)$ and $\vec{\pi}(\tau)$ by means of the gauge fixings $\vec{\mathcal{K}} \stackrel{def}{=} -M \vec{R}_+ \approx 0$. For two *free* particles Eqs.(2.14) imply $(\vec{\eta}(\tau) \approx 0$ for $m_1 = m_2)$

$$\vec{\eta}_+(\tau) \approx \vec{\eta}(\tau) = \frac{\frac{m_1}{m} \sqrt{m_2^2 c^2 + \vec{\pi}^2(\tau)} - \frac{m_2}{m} \sqrt{m_1^2 c^2 + \vec{\pi}^2(\tau)}}{\sqrt{m_1^2 c^2 + \vec{\pi}^2(\tau)} + \sqrt{m_2^2 c^2 + \vec{\pi}^2(\tau)}} \vec{\rho}(\tau). \tag{2.16}$$

In the interacting case the rest-frame conditions $\vec{\kappa}_+ \approx 0$ and the conditions eliminating the internal 3-center of mass $\vec{\mathcal{K}} \approx 0$ will determine $\vec{\eta}_+$ in terms of the relative variables $\vec{\rho}$, $\vec{\pi}$ in an interaction-dependent way.

Then the relative variables satisfy Hamilton equations with the invariant mass $M(\vec{\rho}, \vec{\pi})$ as Hamiltonian and the particle world-lines $x_i^\mu(\tau)$ can be rebuilt [7].

The position of the two positive-energy particles in each instantaneous Wigner 3-space is identified by the intersection of the world-lines ($m_3 \equiv m_1$)

$$\begin{aligned}
x_i^\mu(\tau) &= Y^\mu(\tau) + \epsilon_r^\mu(\vec{h}) \eta_i^r(\tau) \approx Y^\mu(\tau) + \epsilon_r^\mu(\vec{h}) \left[\eta^r[\vec{\rho}(\tau), \vec{\pi}(\tau)] + (-)^{i+1} \frac{m_{i+1}}{m} \rho^r(\tau) \right] \\
&\approx_{free\ case} Y^\mu(\tau) + \epsilon_r^\mu(\vec{h}) \frac{\sqrt{m_i^2 c^2 + \vec{\pi}^2(\tau)}}{\sqrt{m_1^2 c^2 + \vec{\pi}^2(\tau)} + \sqrt{m_2^2 c^2 + \vec{\pi}^2(\tau)}} \rho^r(\tau), \\
\vec{x}_i(\tau) &\rightarrow_{c \rightarrow \infty} \vec{x}_{(n)}(t) + (-)^{i+1} \frac{m_{i+1}}{m} \vec{r}_{(n)}(t) = \vec{x}_{(n)i}(t),
\end{aligned}$$

$$p_i^\mu(\tau) = h^\mu \sqrt{m_i^2 c^2 + \vec{\pi}^2(\tau)} + (-)^{i+1} \epsilon_r^\mu(\vec{h}) \pi^r(\tau), \quad \epsilon p_i^2(\tau) = m_i^2 c^2, \quad (2.17)$$

with $Y^\mu(\tau)$ given in Eq.(2.6) in terms of \vec{z} , \vec{h} and τ . In the non-relativistic limit they identify the Newton trajectories $\vec{x}_{(n)i}(t)$ ($\vec{x}_{(n)}$ and $\vec{r}_{(n)}$ are the center-of-mass and relative variable, respectively). The covariant predictive world-lines $x_i^\mu(\tau)$ depend on the relative position variables $\vec{\rho}$ and we get that:

a) if the interaction among the particles is such that the relative position variables have a compact support when τ varies (as happens with the classical analogue of bound states) the world-lines will be included in some finite time-like world-tube;

b) instead, if the interactions describe the classical analogue of scattering states, the world-lines can diverge one from the other (cluster decomposition property).

This qualitative description has to be checked in every system with a well defined action-at-a-distance interaction.

They turn out to have a non-commutative (predictive) associated structure given in Eq.(2.14) of Ref.[1].

B. Collective and Relative Variables for N Particles

Let us introduce collective and relative variables on the Wigner hyper-planes and eliminate the collective ones by using the rest-frame conditions $\vec{\mathcal{P}} \approx 0$ and $\vec{\mathcal{K}} \approx 0$.

In previous papers [6, 7], [4], the problem of replacing the 3-coordinates $\vec{\eta}_i(\tau)$, $\vec{\kappa}_i(\tau)$ inside a Wigner hyper-plane with internal collective and relative canonical variables in the rest-frame instant form was solved in two different ways:

a) We may introduce naive collective variables $\vec{\eta}_+ = \frac{1}{N} \sum_i^{1..N} \vec{\eta}_i$ (independent of the particle masses), $\vec{\kappa}_+ = \vec{\mathcal{P}} = \sum_i^{1..N} \vec{\kappa}_i \approx 0$ and then completing them with either naive relative variables $\vec{\rho}_a = \sqrt{N} \sum_{i=1}^N \gamma_{ai} \vec{\eta}_i$, $\vec{\pi}_a = \frac{1}{\sqrt{N}} \sum_{i=1}^N \gamma_{ai} \vec{\kappa}_i$, $a = 1, \dots, N-1$, or with the relative variables in the so-called spin bases¹⁷. This naive canonical basis is obtained with a linear canonical transformation *point* both in the positions and in the momenta.

¹⁷ Both sets can be used to find the expression of the Dixon multipoles for a two-particle open subsystem in terms of canonical c.o.m and relative canonical variables [6].

b) We may find the canonical basis whose collective variables are the internal 3-center of mass \vec{q}_+ and $\vec{\kappa}_+ = \vec{\mathcal{P}} = \sum_i^{1..N} \vec{\kappa}_i \approx 0$. To these collective variables are then associated relative variables $\vec{\rho}_{qa}, \vec{\pi}_{qa}, a = 1, \dots, N-1$. However this non-linear canonical transformation depends upon the interactions present among the particles (through the internal Poincare' generators), so that it is known *only for free particles*: even in this case it is *point only in the momenta*. The advantage of this canonical transformation would be to allow one to write the invariant mass in the form $Mc = \sqrt{\mathcal{M}^2 c^2 + \vec{\kappa}_+^2} \approx \mathcal{M} c$ ¹⁸, explicitly showing that in the rest-frame the internal mass depends only on relative variables.

Since it is more convenient to use the naive linear canonical transformation we will use the following collective and relative variables which, written in terms of the masses of the particles, make it easier to evaluate the non-relativistic limit ($m = \sum_{i=1}^N m_i$)

$$\begin{aligned}
\vec{\eta}_+ &= \sum_{i=1}^N \frac{m_i}{m} \vec{\eta}_i, & \vec{\kappa}_+ = \vec{\mathcal{P}} &= \sum_{i=1}^N \vec{\kappa}_i, \\
\vec{\rho}_a &= \sqrt{N} \sum_{i=1}^N \gamma_{ai} \vec{\eta}_i, & \vec{\pi}_a &= \frac{1}{\sqrt{N}} \sum_{i=1}^N \Gamma_{ai} \vec{\kappa}_i, & a &= 1, \dots, N-1, \\
\vec{\eta}_i &= \vec{\eta}_+ + \frac{1}{\sqrt{N}} \sum_{a=1}^{N-1} \Gamma_{ai} \vec{\rho}_a, \\
\vec{\kappa}_i &= \frac{m_i}{m} \vec{\kappa}_+ + \sqrt{N} \sum_{a=1}^{N-1} \gamma_{ai} \vec{\pi}_a,
\end{aligned} \tag{2.18}$$

with the following canonicity conditions¹⁹

$$\begin{aligned}
\sum_{i=1}^N \gamma_{ai} &= 0, & \sum_{i=1}^N \gamma_{ai} \gamma_{bi} &= \delta_{ab}, & \sum_{a=1}^{N-1} \gamma_{ai} \gamma_{aj} &= \delta_{ij} - \frac{1}{N}, \\
\Gamma_{ai} &= \gamma_{ai} - \sum_{k=1}^N \frac{m_k}{m} \gamma_{ak}, & \gamma_{ai} &= \Gamma_{ai} - \frac{1}{N} \sum_{k=1}^N \Gamma_{ak}, \\
\sum_{i=1}^N \frac{m_i}{m} \Gamma_{ai} &= 0, & \sum_{i=1}^N \gamma_{ai} \Gamma_{bi} &= \delta_{ab}, & \sum_{a=1}^{N-1} \gamma_{ai} \Gamma_{aj} &= \delta_{ij} - \frac{m_i}{m}.
\end{aligned} \tag{2.19}$$

For $N = 2$ we have $\gamma_{11} = -\gamma_{12} = \frac{1}{\sqrt{2}}$, $\Gamma_{11} = \sqrt{2} \frac{m_2}{m}$, $\Gamma_{12} = -\sqrt{2} \frac{m_1}{m}$.

¹⁸ For two free particles we have $Mc = \sqrt{m_1^2 c^2 + \vec{\kappa}_1^2} + \sqrt{m_2^2 c^2 + \vec{\kappa}_2^2} = \sqrt{\mathcal{M}^2 c^2 + \vec{\kappa}_+^2} \approx \mathcal{M} c = \sqrt{m_1^2 c^2 + \vec{\pi}_q^2} + \sqrt{m_2^2 c^2 + \vec{\pi}_q^2}$.

¹⁹ Eqs.(2.18) describe a family of canonical transformations, because the γ_{ai} 's depend on $\frac{1}{2}(N-1)(N-2)$ free independent parameters.

When the internal boost has the form $\vec{K} = -\sum_i \vec{\eta}_i E_i$ like in the free case (where $E_i = \sqrt{m_i^2 c^2 + \vec{\kappa}_i^2}$) the gauge fixings $\vec{K} \approx 0$ to the rest-frame conditions $\vec{\mathcal{P}} \approx 0$ imply

$$\vec{\eta}_+(\tau) \approx \vec{\eta}(\vec{\rho}_b(\tau), \vec{\pi}_b(\tau)) = -\frac{1}{\sqrt{N}} \sum_a \frac{\sum_i \Gamma_{ai} E_i(\tau)}{Mc} \vec{\rho}_a(\tau). \quad (2.20)$$

C. More on N-Body Systems

As already said our description of isolated N-particle systems, induced by the need to eliminate time-like oscillations in relativistic bound states, is based on radar 4-coordinates $\sigma^A = (\tau; \sigma^r)$ centered on the Fokker-Pryce center of inertia $Y^\mu(\tau)$ and separates the decoupled (non-local, non-covariant, non-measurable) relativistic center of mass, which is parametrized in terms of the frozen (non-evolving) Jacobi data \vec{z}, \vec{h} .

1. N Particles in the Wigner 3-Spaces Σ_τ with $\tau = \text{const.}$

In an arbitrary inertial frame in Minkowski space-time, where the Fokker-Pryce center of inertia and the embedding of the Wigner 3-spaces are given by Eqs. (2.6) and (2.3) respectively, the derived world-lines (2.11) of the particles have the expression

$$\begin{aligned} x_i^o(\tau) &= \sqrt{1 + \vec{h}^2} \left(\tau + \frac{\vec{h} \cdot \vec{z}}{Mc} \right) + \vec{h} \cdot \vec{\eta}_i(\tau), \\ \vec{x}_i(\tau) &= \frac{\vec{z}}{Mc} + \vec{\eta}_i(\tau) + \frac{\vec{S} \times \vec{h}}{1 + \sqrt{1 + \vec{h}^2}} + \\ &\quad + \vec{h} \left(\tau + \frac{\vec{h} \cdot \vec{z}}{Mc} + \frac{\vec{h} \cdot \vec{\eta}_i(\tau)}{1 + \sqrt{1 + \vec{h}^2}} \right), \\ x_i^o(0) &= \sqrt{1 + \vec{h}^2} \frac{\vec{h} \cdot \vec{z}}{Mc} + \vec{h} \cdot \vec{\eta}_i(0), \\ \vec{x}_i(0) &= \frac{\vec{z}}{Mc} + \vec{\eta}_i(0) + \frac{\vec{S} \times \vec{h}}{1 + \sqrt{1 + \vec{h}^2}} + \\ &\quad + \vec{h} \left(\frac{\vec{h} \cdot \vec{z}}{Mc} + \frac{\vec{h} \cdot \vec{\eta}_i(0)}{1 + \sqrt{1 + \vec{h}^2}} \right). \end{aligned} \quad (2.21)$$

The inversion of Eqs.(2.21) gives the following form of $\eta_i^A(\tau) = (\tau; \eta_i^r(\tau))$

$$\begin{aligned} \tau &= \sqrt{1 + \vec{h}^2} x_i^o(\tau) - \vec{h} \cdot \vec{x}_i(\tau), \\ \vec{\eta}_i(\tau) &= \vec{x}_i(\tau) - \frac{\vec{z}}{Mc} - \frac{\vec{S} \times \vec{h}}{1 + \sqrt{1 + \vec{h}^2}} + \\ &\quad + \vec{h} \left(\vec{h} \cdot \frac{\vec{x}_i(\tau) - \frac{\vec{z}}{Mc}}{1 + \sqrt{1 + \vec{h}^2}} - \frac{x_i^o(\tau)}{\sqrt{1 + \vec{h}^2}} \right). \end{aligned} \quad (2.22)$$

We also have $\vec{h} \cdot \vec{\eta}_i(\tau) = \sqrt{1 + \vec{h}^2} \vec{h} \cdot \left(\vec{x}_i(\tau) - \frac{\vec{z}}{Mc} \right) - \vec{h}^2 x_i^o(\tau)$.

For the derived (interaction-dependent) momenta (2.12) we have

$$\begin{aligned} p_i^o(\tau) &= \sqrt{1 + \vec{h}^2} E_i(\tau) + \vec{h} \cdot \vec{\kappa}_i(\tau), \\ \vec{p}_i(\tau) &= \vec{\kappa}_i(\tau) + \vec{h} \left(E_i(\tau) + \frac{\vec{h} \cdot \vec{\kappa}_i(\tau)}{1 + \sqrt{1 + \vec{h}^2}} \right). \end{aligned} \quad (2.23)$$

In the free case the inversion of Eqs.(2.23) to get $\kappa_i^A(\tau) = (E_i(\tau); \kappa_{ir}(\tau))$ is

$$\begin{aligned} E_i(\tau) &= \sqrt{m_i^2 c^2 + \vec{\kappa}_i^2(\tau)} = \sqrt{1 + \vec{h}^2} p_i^o(\tau) - \vec{h} \cdot \vec{p}_i(\tau), \\ \vec{\kappa}_i(\tau) &= \vec{p}_i(\tau) - \vec{h} \left(p_i^o(\tau) - \frac{\vec{h} \cdot \vec{p}_i(\tau)}{1 + \sqrt{1 + \vec{h}^2}} \right). \end{aligned} \quad (2.24)$$

We also have $\vec{h} \cdot \vec{\kappa}_i(\tau) = \sqrt{1 + \vec{h}^2} \vec{h} \cdot \vec{p}_i(\tau) - \vec{h}^2 p_i^o(\tau)$, $Mc = \sqrt{1 + \vec{h}^2} \sum_i p_i^o - \vec{h} \cdot \sum_i \vec{p}_i$ and $e^{-\beta E_i} = e^{-\beta_\mu p_i^\mu}$ ($\beta = 1/k_B T$, $\beta^\mu = \beta h^\mu$).

In the free case the Hamilton equations with Mc as Hamiltonian have the solutions

$$\begin{aligned} \vec{k}_i &= \frac{m_i c \dot{\vec{\eta}}_i}{\sqrt{1 - \dot{\vec{\eta}}_i^2}} = \text{const}, \quad \dot{\vec{\eta}}_i = \text{const.}, \\ \vec{\eta}_i(\tau) &= \vec{\eta}_i(0) + \frac{\vec{\kappa}_i}{\sqrt{m_i^2 c^2 + \vec{\kappa}_i^2}} \tau = \vec{\eta}_i(0) + \dot{\vec{\eta}}_i \tau, \end{aligned} \quad (2.25)$$

with $\vec{\kappa}_i$ and $\vec{\eta}_i(0)$ restricted by $\vec{\mathcal{P}} \approx 0$ and $\vec{\mathcal{K}} \approx 0$.

In this case Eqs.(2.21) and (2.23) imply

$$\begin{aligned} x_i^o(\tau) &= x_i^o(0) + \left(\sqrt{1 + \vec{h}^2} + \frac{\vec{h} \cdot \vec{\kappa}_i}{\sqrt{m_i^2 c^2 + \vec{\kappa}_i^2}} \right) \tau = x_i^o(0) + \frac{p_i^o}{E_i} \tau, \\ \vec{x}_i(\tau) &= \vec{x}_i(0) + \left[\frac{\vec{\kappa}_i}{\sqrt{m_i^2 c^2 + \vec{\kappa}_i^2}} + \vec{h} \left(1 + \frac{\vec{h} \cdot \vec{k}_i / \sqrt{m_i^2 c^2 + \vec{\kappa}_i^2}}{1 + \sqrt{1 + \vec{h}^2}} \right) \right] \tau, \\ \Rightarrow \quad \vec{x}_i(\tau) - \vec{x}_i(0) &= \frac{\vec{p}_i}{p_i^o} \left(x_i^o(\tau) - x_i^o(0) \right). \end{aligned} \quad (2.26)$$

In the inertial rest frame with Cartesian coordinates $x_{(cm)}^\mu$ and with $\vec{h} = 0$, where $P_{(cm)}^\mu = \sum_i p_{(cm)i}^\mu(\tau) = Mc h_{(cm)}^\mu = Mc(1; 0)$ and $Y_{(cm)}^\mu(\tau) = \left(\tau; \frac{\vec{z}_{(cm)}}{Mc} \right)$ from Eq. (2.6), from Eq.(2.3) we get $x_{(cm)}^o = \tau$, $\vec{x}_{(cm)} = \vec{\sigma}$, and Eqs. (2.21), (2.18), (2.20), imply

$$\begin{aligned}
x_{(cm)i}^o(\tau) &= \tau = x_{(cm)}^o, & p_{(cm)i}^o(\tau) &= E_i(x_{(cm)}^o), \\
\vec{x}_{(cm)i}(\tau) &= \frac{\vec{z}_{(cm)}}{Mc} + \vec{\eta}_i(\tau) \approx \\
&\approx \frac{\vec{z}_{(cm)}}{Mc} + \vec{\eta}(\vec{\rho}_b(\tau), \vec{\pi}_b(\tau)) + \frac{1}{\sqrt{N}} \sum_a \Gamma_{ai} \vec{\rho}_a(\tau) = \\
&= \vec{x}_{(cm)i}(x_{(cm)}^o) = \vec{x}_{(cm)+}(x_{(cm)}^o) + \frac{1}{\sqrt{N}} \sum_a \Gamma_{ai} \vec{\rho}_a(x_{(cm)}^o), \\
\vec{x}_{(cm)+}(x_{(cm)}^o) &= \frac{\vec{z}}{Mc} + \vec{\eta}(\vec{\rho}_b(x_{(cm)}^o), \vec{\pi}_b(x_{(cm)}^o)), \\
\vec{p}_{(cm)i}(\tau) &= \vec{\kappa}_i(\tau) \approx \sqrt{N} \sum_a \gamma_{ai} \vec{\pi}_a(x_{(cm)}^o) = \vec{p}_{(cm)i}(x_{(cm)}^o).
\end{aligned} \tag{2.27}$$

In conclusion in the inertial rest frame the N-body system is described by the following 6N+1 variables: a) the proper time τ of the inertial observer corresponding to the Fokker-Pryce center of inertia of the isolated system; b) the frozen (non-evolving) Jacobi data \vec{z} and \vec{h} of the external decoupled (non-covariant, non-local, non measurable) center of mass; c) the relative (Wigner spin-1) 3-vectors $\vec{\rho}_a(\tau)$, $\vec{\pi}_a(\tau)$, $a = 1, \dots, N-1$.

2. N Particles in the 3-Spaces Σ_{x^o} with $x^o = \text{const.}$

For the arbitrary inertial observer with Cartesian orthogonal 4-coordinates $x^\mu = (x^o; \vec{x})$ the point with radar 4-coordinates $\sigma^A = (\tau; \vec{\sigma})$ inside the Wigner 3-space Σ_τ has coordinates $x^o(\tau, \vec{\sigma}) = z_W^o(\tau, \vec{\sigma}) = Y^o(0) + \sqrt{1 + \vec{h}^2} \tau + \vec{h} \cdot \vec{\sigma}$, $\vec{x}(\tau, \vec{\sigma}) = \vec{z}_W(\tau, \vec{\sigma}) = \vec{Y}(0) + \vec{\sigma} + \vec{h} \left(\tau + \frac{\vec{h} \cdot \vec{\sigma}}{1 + \sqrt{1 + \vec{h}^2}} \right)$ with $Y^\mu(0)$ given in Eq.(2.6).

Instead the points $x^\mu = (x^o; \vec{x})$ inside the Euclidean 3-space Σ_{x^o} with $x^o = \text{const.}$ have radar 4-coordinates

$$\begin{aligned}
\tau(x^o; \vec{x}) &= \sqrt{1 + \vec{h}^2} (x^o - Y^o(0)) - \vec{h} \cdot (\vec{x} - \vec{Y}(0)), \\
\vec{\sigma}(x^o, \vec{x}) &= \vec{x} - \vec{Y}(0) - \vec{h} \left(x^o - Y^o(0) - \frac{\vec{h} \cdot (\vec{x} - \vec{Y}(0))}{1 + \sqrt{1 + \vec{h}^2}} \right).
\end{aligned} \tag{2.28}$$

The straight trajectory $Y^\mu(\tau) = Y^\mu(0) + h^\mu \tau$ of the Fokker-Pryce center of inertia (origin of the radar 3-coordinates) can be parametrized in terms of x^o in the following way. By putting $Y^o(\tau) = x^o$ we get $\tau(x^o) = \frac{x^o - Y^o(0)}{\sqrt{1 + \vec{h}^2}} \stackrel{\text{def}}{=} \tau_{FP}(x^o)$ (so that $\frac{d\tau_{FP}(x^o)}{dx^o} = \frac{1}{\sqrt{1 + \vec{h}^2}}$). Then, by using Eq.(2.6), the new parametrization of the Fokker-Planck center of inertia is

$$\mathcal{Y}^\mu(x^o) = \left(x^o; \vec{\mathcal{Y}}(x^o)\right), \quad \mathcal{Y}^\mu(x^o = Y^o(0)) = Y^\mu(0),$$

$$\vec{\mathcal{Y}}(x^o) = \vec{Y}(\tau_{FP}(x^o)) = \frac{\vec{z}}{Mc} + \frac{\vec{h}}{\sqrt{1 + \vec{h}^2}} x^o + \frac{\vec{S} \times \vec{h}}{Mc(1 + \sqrt{1 + \vec{h}^2})}. \quad (2.29)$$

Let us remark that if we choose the origin of the inertial frame such that $Y^\mu(0) = 0$ then we have $\vec{z} = -\frac{\vec{S} \times \vec{h}}{Mc(1 + \sqrt{1 + \vec{h}^2})}$ with $\vec{h} \cdot \vec{z} = 0$.

Let us now consider the world-lines of the particles and the associated momenta. By using Eq.(2.21) it is clear that in the Euclidean 3-space Σ_{x^o} the position of each particle has to be evaluated at a different proper time $\tau_i(x^o)$ solution of the equations

$$x^o = x_i^o(\tau_i) = \sqrt{1 + \vec{h}^2} \tau_i + Y^o(0) + \vec{h} \cdot \vec{\eta}_i(\tau_i). \quad (2.30)$$

Let us remark that to find the solution $\tau_i(x^o)$ of Eqs.(2.30) we need the solution $\vec{\eta}_i(\tau)$, $\vec{\kappa}_i(\tau)$ of the Hamilton equations with Hamiltonian Mc . Then we get the following new parametrization of the world-lines

$$\begin{aligned} \tilde{x}_i^\mu(x^o) &= \left(x^o; \vec{\tilde{x}}_i(x^o)\right) = x_i^\mu(\tau_i(x^o)) = Y^\mu(0) + \Lambda^\mu_A(\vec{h}) \tilde{\eta}_i^a(x^o), \\ \tilde{\eta}_i^A(x^o) &= \eta_i^A(\tau_i(x^o)) = \left(\tau_i(x^o); \vec{\tilde{\eta}}_i(x^o) = \vec{\eta}_i(\tau_i(x^o))\right), \\ \vec{\tilde{x}}_i(x^o) &= \frac{\vec{z}}{Mc} + \vec{\tilde{\eta}}_i(x^o) + \frac{\vec{S} \times \vec{h}}{Mc(1 + \sqrt{1 + \vec{h}^2})} + \frac{\vec{h}}{\sqrt{1 + \vec{h}^2}} \left(x^o - \frac{\vec{h} \cdot \vec{\tilde{\eta}}_i(x^o)}{1 + \sqrt{1 + \vec{h}^2}}\right). \end{aligned} \quad (2.31)$$

Then Eqs.(2.23) imply the following parametrization of the momenta

$$\begin{aligned} \tilde{p}_i^o(x^o) &= p_i^o(\tau_i(x^o)) = \sqrt{1 + \vec{h}^2} \tilde{E}_i(x^o) + \vec{h} \cdot \vec{\tilde{\kappa}}_i(x^o), \\ \vec{\tilde{p}}_i(x^o) &= \vec{p}_i(\tau + i(x^o)) = \vec{\tilde{\kappa}}_i(x^o) + \vec{h} \left(\tilde{E}_i(x^o) + \frac{\vec{h} \cdot \vec{\tilde{\kappa}}_i(x^o)}{1 + \sqrt{1 + \vec{h}^2}} \right), \end{aligned} \quad (2.32)$$

where $\vec{\tilde{\kappa}}_i(x^o) = \vec{\kappa}_i(\tau_i(x^o))$ and $\tilde{E}_i(x^o) = E_i(\tau_i(x^o))$. The kinematical first half of the Hamilton equations with Hamiltonian Mc imply $\frac{dx_i^o(\tau_i)}{d\tau_i} = \frac{\tilde{p}_i^o(x^o)}{\tilde{E}_i(x^o)}$. Eqs.(2.31) and (2.32) can be obtained explicitly only after having solved the Hamilton equations for $\vec{\eta}_i(\tau)$ and $\vec{\kappa}_i(\tau)$.

In the free case we have $\vec{\eta}_i(\tau) = \vec{\eta}_i(0) + \frac{\vec{\kappa}_i}{E_i} \tau$, $\vec{\kappa}_i(\tau) = \vec{\kappa}_i = \text{const.}$ and $E_i(\tau) = E_i = \sqrt{m_i^2 c^2 + \vec{\kappa}_i^2} = \text{const.}$. Then the solution of Eqs.(2.30) is

$$\tau_i(x^o) = \frac{x^o - Y^o(0) - \vec{h} \cdot \vec{\eta}_i(0)}{\sqrt{1 + \vec{h}^2}} = \tau_{FP}(x^o) - \frac{\vec{h} \cdot \vec{\eta}_i(0)}{\sqrt{1 + \vec{h}^2}}, \quad (2.33)$$

and Eqs.(2.32) give (one could also replace x^o with $Y^o(0) + \sqrt{1 + \vec{h}^2} \tau_{FP}(x^o)$)

$$\begin{aligned}
\vec{\eta}_i(x^o) &= \vec{\eta}_i(0) + \frac{\vec{\kappa}_i}{E_i \sqrt{1 + \vec{h}^2}} x^o, \\
\vec{\eta}_i(x^o = 0) &= \vec{\eta}_i(\tau = 0) - \frac{\vec{\kappa}_i}{E_i} \left(Y^o(0) + \vec{h} \cdot \vec{\eta}_i(\tau = 0) \right), \\
\vec{x}_i(x^o) &= \vec{x}_i(0) + \left[\frac{\vec{\kappa}_i}{E_i} + \vec{h} \left(1 - \frac{\vec{h} \cdot \vec{\kappa}_i}{E_i \sqrt{1 + \vec{h}^2} (1 + \sqrt{1 + \vec{h}^2})} \right) \right] \frac{x^o}{\sqrt{1 + \vec{h}^2}}, \\
\vec{x}_i(0) &= \frac{\vec{z}}{Mc} + \frac{\vec{S} \times \vec{h}}{Mc (1 + \sqrt{1 + \vec{h}^2})} + \vec{\eta}_i(0) - \frac{\vec{h}}{\sqrt{1 + \vec{h}^2}} \frac{\vec{h} \cdot \vec{\eta}_i(0)}{1 + \sqrt{1 + \vec{h}^2}}. \quad (2.34)
\end{aligned}$$

Therefore the transition from the $6N+1$ variables $\tau, \vec{z}, \vec{h}, \vec{\rho}_a(\tau), \vec{\pi}_a(\tau), a = 1, \dots, N-1$, in the rest frame Wigner 3-space Σ_τ to the $6N+1$ variables $x^o, \vec{x}_i(x^o), \vec{p}_i(x^o)$, in the Euclidean 3-space Σ_{x^o} can be done *only on-shell*, i.e. on the solution of the Hamilton equations.

Finally, if Λ is the Lorentz transformation from the rest frame with coordinates $x_{(cm)}^\mu = (\tau; \vec{\sigma})$ and Wigner 3-space Σ_τ to the inertial frame with Cartesian coordinates x^μ and Euclidean 3-spaces Σ_{x^o} , we have

$$\begin{aligned}
h^\mu &= (\sqrt{1 + \vec{h}^2}; \vec{h}) = \Lambda^\mu{}_\nu h_{(cm)}^\nu, \\
\tilde{x}_i^\mu(x^o) &= x_i^\mu(\tau_i(x^o)) = \Lambda^\mu{}_\nu x_{(cm)i}^\nu(\tau = x_{(cm)}^o), \\
\tilde{p}_i^\mu(x^o) &= p_i^\mu(\tau_i(x^o)) = \Lambda^\mu{}_\nu p_{(cm)i}^\nu(\tau = x_{(cm)}^o). \quad (2.35)
\end{aligned}$$

In this inertial frame the Fokker-Pryce center of inertia is $\mathcal{Y}^\mu(x^o) = \Lambda^\mu{}_\nu Y_{(cm)}^\nu(\tau = x_{(cm)}^o) = Y^\mu(\tau)$. While $Y_{(cm)}^\mu(\tau = x_{(cm)}^o)$ depends on the non-covariant Jacobi data $\vec{z}_{(cm)}$, its form $Y^\mu(\tau)$ depends upon the Jacobi data \vec{z} of that frame according to Eq.(2.6) ²⁰.

D. The Non-Relativistic Limit of the Inertial Rest Frame

As shown in Ref.[1] for the case $N = 2$, (easily extended to arbitrary N), the non-relativistic limit of the inertial rest frame leads to the description of N non-relativistic particles in Galilei space-time in terms of the Newton center of mass $\vec{x}_{(n)}(t)$, with conjugate momentum $\vec{p}_{(n)}$, and of relative variables $\vec{\rho}_{(n)a}(t), \vec{\pi}_{(n)}(t), a = 1, \dots, N-1$ like in Eqs.(2.18). The non-relativistic rest frame is defined with the conditions $\vec{x}_{(n)}(t) \approx 0$ and $\vec{p}_{(n)} \approx 0$. Then one can reformulate the theory in terms of the Newton positions $\vec{x}_{(n)i}(t)$ and momenta $\vec{p}_{(n)i}(t)$ of the particles.

By putting $\tau = ct$ and by using Eqs. (2.5), (2.6), (2.7), (2.11), one can show that $\tilde{x}^o/c, Y^o/c, R^o/c, x_i^o/c$, all tend to the absolute Newton time t for $c \rightarrow \infty$. Moreover one has:

²⁰ This is due to the transformation properties $z^i \mapsto z'^i = \left(\Lambda^i{}_j - \frac{\Lambda^i{}_\alpha h^\alpha}{\Lambda^o{}_\beta h^\beta} \Lambda^o{}_j \right) z^j$, see Eq.(2.5) of Ref.[1], and is connected with the fact that the relativistic center of mass $\tilde{x}^\mu(\tau)$ of Eq.(2.5) is not a 4-vector (in each inertial frame it is described by a different pseudo-world-line).

$\vec{x}(\tau)$, $\vec{Y}(\tau)$, $\vec{R}(\tau) \rightarrow_{c \rightarrow \infty} \vec{x}_{(n)}(t)$ and $\vec{x}_{NW}(0) = \vec{z}/Mc \rightarrow_{c \rightarrow \infty} \vec{x}_{(n)}(0)$. Therefore the external center of mass and all the relativistic collective variables collapse into the Newton center of mass. For the total momentum we have $\vec{P} = \vec{p}_{(n)}$ and Eqs.(2.4) imply $h^\mu \rightarrow_{c \rightarrow \infty} (1; 0)$, $\epsilon_r^\mu(\vec{h}) \rightarrow_{c \rightarrow \infty} (0; \delta_r^i)$.

The spatial part of Eq.(2.3) becomes $\vec{z}_W(\tau, \vec{\sigma}) \rightarrow_{c \rightarrow \infty} \vec{x}_{(n)}(t) + \vec{\sigma}$: the non-relativistic inertial frame is centered on the Newton center of mass. The spatial part of the world-lines (2.11) becomes $\vec{x}_i(\tau) \rightarrow_{c \rightarrow \infty} \vec{x}_{(n)i}(t) = \vec{x}_{(n)}(t) + \vec{\eta}_i(t)$ (see Eqs.(2.18) with $\vec{\eta}_+ \mapsto \vec{x}_{(n)}$), where $\vec{\eta}_i(t) = \vec{\eta}_i(\tau)$ are restricted by the vanishing of the non-relativistic limit of the internal Lorentz boosts of Eq.(2.9), $\vec{K}/c \rightarrow_{c \rightarrow \infty} -\sum_i m_i \vec{\eta}_i(t) \approx 0$, so that they define positions $\vec{\eta}_{(n)}(t)$ coinciding with $\vec{x}_{(n)i}(t)$ in the non-relativistic rest frame. The particle momenta $\vec{\kappa}_i(\tau) = \vec{\kappa}_i(t)$ collapse into momenta $\vec{\kappa}_{(n)i}(t)$ restricted by the rest-frame condition $\vec{P} \rightarrow_{c \rightarrow \infty} \sum_i \vec{\kappa}_{(n)i}(t) \approx 0$ (see eqs.(2.18) with $\vec{\kappa}_+ \approx 0$), so that they coincide with the non-relativistic momenta $\vec{p}_{(n)i}(t)$ in the non-relativistic rest frame.

The other internal Poincaré generators Mc and \vec{S} (the pole-dipole carried by the external center of mass) of Eq.(2.9) become in the free case ($m = \sum_i m_i$; $\vec{\eta}_{(n)i}(t) = \vec{x}_{(n)i}(t)|_{\vec{x}_{(n)}=\vec{p}_{(n)}=0}$; $\vec{\kappa}_{(n)i}(t) = \vec{p}_{(n)i}(t)|_{\vec{x}_{(n)}=\vec{p}_{(n)}=0}$)

$$\begin{aligned} Mc &\rightarrow_{c \rightarrow \infty} mc + H_{rel}, & H_{rel} &= \sum_i \frac{\vec{\kappa}_{(n)i}^2(t)}{2m_i}, \\ \vec{S} &\rightarrow_{c \rightarrow \infty} \sum_i \vec{\eta}_{(n)i}(t) \times \vec{\kappa}_{(n)i}(t) = \vec{S}_{(n)}. \end{aligned} \quad (2.36)$$

In the interacting case a potential $V(\vec{\eta}_{(n)i}(t) - \vec{\eta}_{(n)j}(t), \vec{\kappa}_{(n)i}(t))$ will be present in the relative Hamiltonian (or internal energy) H_{rel} . Eqs.(2.18) give the connection between the variables $\vec{x}_{(n)i}(t)$, $\vec{p}_{(n)i}(t)$ and $\vec{x}_{(n)}(t)$, $\vec{p}_{(n)}$, $\vec{p}_{(n)a}(t)$, $\vec{\pi}_{(n)a}(t)$.

The non-relativistic limit of the external Poincaré generators (2.8) gives rise to the generators of the (external) Galilei algebra (centrally extended with the total mass m)

$$\begin{aligned} P^0 &\rightarrow_{c \rightarrow \infty} mc + E_{Galilei}, & E_{Galilei} &= \frac{\vec{p}_{(n)}^2}{2m} + \sum_i \frac{\vec{\kappa}_{(n)i}^2(t)^2}{2m_i} = \sum_i \frac{\vec{p}_{(n)i}^2(t)}{2m_i}, \\ \vec{P} &= \vec{p}_{(n)} = \vec{P}_{Galilei}, \\ \vec{J} &= \vec{x}_{(n)}(t) \times \vec{p}_{(n)} + \vec{S}_{(n)} = \sum_i \vec{x}_{(n)i}(t) \times \vec{p}_{(n)i}(t) = \vec{J}_{Galilei}, \\ \frac{1}{c} \vec{K} &\rightarrow_{c \rightarrow \infty} t \vec{p}_{(n)} - m \vec{x}_{(n)} = \vec{K}_{Galilei}. \end{aligned} \quad (2.37)$$

In the non-relativistic rest frame $\vec{p}_{(n)} \approx 0$, $\vec{x}_{(n)} \approx 0$, there is an unfaithful internal Galilei algebra: $E_{int} = H_{rel} = \sum_i \frac{\vec{\kappa}_{(n)i}^2(t)}{2m_i}$, $\sum_i \vec{\kappa}_{(n)i} \approx 0$, $\vec{S}_{(n)}$, $\vec{K}_{(n)} = -\sum_i m_i \vec{\eta}_{(n)i}(t) - m \vec{x}_{(n)} \approx 0$.

This implies that the generators of the external Galilei algebra (2.37) in the rest frame $\vec{p}_{(n)} = 0$, with the origin in the center of mass ($\vec{x}_{(n)} = 0$), become $E_{Galilei} = E_{int} = H_{rel}$, $\vec{P}_{Galilei} = 0$, $\vec{J}_{Galilei} = \vec{S}_{(n)}$, $\vec{K}_{(n)} = 0$. But this is the form which can be obtained in this

frame by means of a canonical transformation implying the Hamilton-Jacobi description of the center of mass ($H_{com} = \frac{\vec{p}_{(n)}^2}{2m} \mapsto H_{com}^{(HJ)} = 0$).

The non-relativistic limit of the relativistic N-body problem reproduces this Hamilton-Jacobi version of the non-relativistic N-body problem [1].

III. EXTENSION TO THE NON-INERTIAL REST FRAMES

Let us extend this construction to an arbitrary admissible non-inertial frame described by suitable embeddings and centered on an arbitrary time-like observer [3] to determine the three pairs of second class constraints eliminating the internal center of mass. Again the isolated system can be visualized as a pole-dipole carried by the external decoupled center of mass. Then the construction will be restricted to the non-inertial rest frame of an isolated system.

A. Admissible Non-Inertial Frames

Let us now see whether in an arbitrary admissible non-inertial frame, centered on an arbitrary non-inertial observer and described by the embeddings ($f^A(\tau)$ identifies the world-line of the observer; $F^A(\tau, \vec{\sigma})$ identifies the 3+1 splitting of space-time)

$$z^\mu(\tau, \vec{\sigma}) = x^\mu(\tau) + F^\mu(\tau, \vec{\sigma}) = x_o^\mu + \epsilon_A^\mu \left[f^A(\tau) + F^A(\tau, \vec{\sigma}) \right], \quad F^\mu(\tau, \vec{0}) = 0,$$

$$x^\mu(\tau) = x_o^\mu + \epsilon_A^\mu f^A(\tau). \quad (3.1)$$

we can arrive at the same picture of an isolated system as a decoupled external canonical non-covariant center of mass \vec{z} , \vec{h} , carrying a pole-dipole structure, with the external Poincare' generators given by expressions like Eqs.(2.8) and with the dynamics described by suitable relative variables after an appropriate elimination of the internal 3-center of mass inside the instantaneous 3-spaces. If this is possible, there will be a new expression for the internal invariant mass M , a new effective spin \vec{S} (supposed to satisfy the Poisson brackets of an angular momentum and such that $J^i = \delta^{im} \epsilon_{mnk} \left(z^n h^k + \tilde{S}^k \right)$) and a new form of the three pairs of second class constraints replacing the expressions given in Eqs.(2.9) for the case of the inertial rest frame centered on the Fokker-Przyce center of inertia.

At spatial infinity $z^\mu(\tau, \sigma^r)$ of Eq.(3.1) must tend in a direction-independent way to a space-like hyper-plane with unit time-like normal $l_{(\infty)}^\mu = \epsilon_\tau^\mu$: this implies $F^\mu(\tau, \vec{\sigma}) \rightarrow \epsilon_{(\infty)r}^\mu \sigma^r$ with the 3 space-like 4-vectors $\epsilon_{(\infty)r}^\mu = \epsilon_r^\mu$ orthogonal to $l_{(\infty)}^\mu$. The asymptotic orthonormal tetrads ϵ_A^μ are associated to asymptotic inertial observers and satisfy $\epsilon_A^\mu \eta_{\mu\nu} \epsilon_B^\nu = \eta_{AB}$.

We must evaluate the Poincare' generators (2.2) by using Eqs.(3.1) and (2.1). By equating the resulting expressions with Eqs.(2.8) we will find the new expression of the invariant mass, of the effective spin and of the second class constraints.

Since the embeddings (3.1) depend on the asymptotic tetrads ϵ_A^μ , we must express them in terms of the tetrads $\epsilon_A^\mu(\vec{h})$ determined by P^μ (whose expression is given in Eq.(2.4)):

$$\epsilon_A^\mu = \Lambda_A^B(\vec{h}) \epsilon_B^\mu(\vec{h}), \quad (3.2)$$

with $\Lambda(\vec{h})$ a Lorentz matrix.

As shown in Ref.[3] the invariant mass and the effective spin carried by the external decoupled center of mass turn out to be ($l^\mu(\tau, \vec{\sigma}) = \epsilon_A^\mu l^A(\tau, \vec{\sigma})$)

$$\begin{aligned}
Mc &\approx \int d^3\sigma \sqrt{\gamma(\tau, \vec{\sigma})} \left[T_{\perp\perp} l^A - T_{\perp s} h^{sr} \partial_r F^A \right] (\tau, \vec{\sigma}) \Lambda_A^\tau(\vec{h}), \\
\tilde{S}^r &\approx \frac{1}{2} \epsilon^{ruv} \int d^3\sigma \sqrt{\gamma(\tau, \vec{\sigma})} \left[F^C(\tau, \vec{\sigma}) \left(T_{\perp\perp} l^D - T_{\perp s} h^{sr} \partial_r F^D \right) (\tau, \vec{\sigma}) - \right. \\
&\quad \left. - F^D(\tau, \vec{\sigma}) \left(T_{\perp\perp} l^C - T_{\perp s} h^{sr} \partial_r F^C \right) (\tau, \vec{\sigma}) \right] \Lambda_C^u(\vec{h}) \Lambda_D^v(\vec{h}). \tag{3.3}
\end{aligned}$$

Moreover the three pairs of second class constraints eliminating the internal center of mass have the form

$$\begin{aligned}
\tilde{\mathcal{P}}^r &= \int d^3\sigma \sqrt{\gamma(\tau, \vec{\sigma})} \left[T_{\perp\perp} l^A - T_{\perp s} h^{sr} \partial_r F^A \right] (\tau, \vec{\sigma}) \Lambda_A^r(\vec{h}) \approx 0, \\
\hat{\mathcal{K}}^r &\approx Mc h^r \left(x_o^o + f^B(\tau) \Lambda_B^C(\vec{h}) \epsilon_C^o(\vec{h}) - \frac{\sum_u h^u \left(x_o^u - z^u + f^B(\tau) \Lambda_B^C(\vec{h}) \epsilon_C^u(\vec{h}) \right)}{1 + \sqrt{1 + \vec{h}^2}} \right) - \\
&\quad - \left(x_o^r - z^r + f^B(\tau) \Lambda_B^C(\vec{h}) \epsilon_C^r(\vec{h}) + \frac{\delta^{rm} \epsilon_{mnk} h^n \tilde{S}^k}{Mc (1 + \sqrt{1 + \vec{h}^2})} \right). \tag{3.4}
\end{aligned}$$

where $\hat{\mathcal{K}}^r = \int d^3\sigma \sqrt{\gamma(\tau, \vec{\sigma})} \left[F^C(\tau, \vec{\sigma}) \left(T_{\perp\perp} l^D - T_{\perp s} h^{sr} \partial_r F^D \right) (\tau, \vec{\sigma}) - F^D(\tau, \vec{\sigma}) \left(T_{\perp\perp} l^C - T_{\perp s} h^{sr} \partial_r F^C \right) (\tau, \vec{\sigma}) \right] \Lambda_C^r(\vec{h}) \Lambda_D^\tau(\vec{h})$.

Let us remark that that if we put $\Lambda_A^B(\vec{h}) = \delta_A^B$ and $x_o^\mu + f^B(\tau) \Lambda_B^C(\vec{h}) \epsilon_C^\mu(\vec{h}) = Y^\mu(0) + h^\mu \tau$, then we recover the results of Subsection A of Section II for the inertial rest frame centered on the Fokker-Pryce inertial observer when $F^A(\tau, \vec{\sigma}) = \sigma^A$.

Instead the conditions $\Lambda_A^B(\vec{h}) = \delta_A^B$ and $f^B(\tau) \Lambda_B^C(\vec{h}) \epsilon_C^\mu(\vec{h}) = h^\mu \tau$, identifying the inertial rest frame centered on the inertial observer $x_o^\mu + h^\mu \tau$, have the constraints $\mathcal{K}^r \approx 0$ replaced by the second of Eqs.(3.4).

Equations of the type (3.3) and (3.4) holds not only for admissible embeddings with pure differential rotations ($\sigma = |\vec{\sigma}|$)

$$z^\mu(\tau, \vec{\sigma}) = x^\mu(\tau) + \epsilon_\tau^\mu R_s^r(\tau, \sigma) \sigma^s, \quad x^\mu(\tau) = x_o^\mu + f^A(\tau) \epsilon_A^\mu,$$

$$R_s^r(\tau, \sigma) = R_s^r(\alpha_i(\tau, \sigma)) = R_s^r(F(\sigma) \tilde{\alpha}_i(\tau)),$$

$$0 < F(\sigma) < \frac{1}{A\sigma}, \quad \frac{dF(\sigma)}{d\sigma} \neq 0 \text{ (Moller conditions)}, \tag{3.5}$$

but also for the admissible embeddings with pure linear acceleration. If in Eq.(2.1) we put $F^\tau(\tau, \vec{\sigma}) = 0$, $F^r(\tau, \vec{\sigma}) = \sigma^r$, so that the embedding becomes $z^\mu(\tau, \vec{\sigma}) = x_o^\mu + \epsilon_\tau^\mu f^\tau(\tau) +$

$\epsilon_r^\mu \left(f^r(\tau) + \sigma^r \right)$, the instantaneous 3-spaces are space-like hyper-planes orthogonal to $l^\mu = \epsilon_\tau^\mu$ and we get $h_{rs} = \delta_{rs}$, $1 + n(\tau) = \dot{f}^r(\tau)$, $n_r(\tau) = \delta_{rs} \dot{f}^s(\tau)$. In the case of the embedding

$$z^\mu(\tau, \vec{\sigma}) = x_o^\mu + \epsilon_\tau^\mu f(\tau) + \epsilon_r^\mu \sigma^r,$$

$$g_{\tau\tau}(\tau, \vec{\sigma}) = \epsilon \left(\frac{df(\tau)}{d\tau} \right)^2, \quad g_{\tau r}(\tau, \vec{\sigma}) = 0, \quad g_{rs}(\tau, \vec{\sigma}) = -\epsilon \delta_{rs}, \quad (3.6)$$

i.e. $f^r(\tau) = 0$ and $f^\tau(\tau) = f(\tau)$, we get $1 + n(\tau) = \dot{f}(\tau)$, $n_r = 0$. If $f^\tau(\tau) = \tau$ and $f^r(\tau) = a^r = \text{const.}$, we have inertial frames centered on inertial observers: changing a^r we change the inertial observer origin of the 3-coordinates σ^r .

In Ref.[3] it is shown that in non-inertial frames the final Dirac Hamiltonian does not coincide with Mc like in Eqs.(2.10) due to the presence of the inertial potentials $g_{AB}(\tau, \sigma^u)$.

B. The Non-Inertial Rest Frames

The family of non-inertial rest frames for an isolated system consists of all the admissible 3+1 splittings of Minkowski space-time whose instantaneous 3-spaces Σ_τ tend to space-like hyper-planes orthogonal to the conserved 4-momentum of the isolated system at spatial infinity. Therefore they tend to the Wigner 3-spaces (2.3) of the inertial rest frame asymptotically. They are relevant because they are the only global non-inertial frames allowed by the equivalence principle (forbidding the existence of global inertial frames) in canonical metric and tetrad gravity [19], [17], in globally hyperbolic, asymptotically flat (asymptotically Minkowskian) space-times without super-translations, so to have the asymptotic ADM Poincare' group.

These non-inertial frames can be centered on the external Fokker-Pryce center of inertia like the inertial ones and are described by the following embeddings (the admissibility conditions are restrictions on the functions $g(\tau, \vec{\sigma})$ and $g^r(\tau, \vec{\sigma})$)

$$\begin{aligned} z^\mu(\tau, \vec{\sigma}) &\approx z_F^\mu(\tau, \vec{\sigma}) = Y^\mu(\tau) + u^\mu(\vec{h}) g(\tau, \vec{\sigma}) + \epsilon_r^\mu(\vec{h}) [\sigma^r + g^r(\tau, \vec{\sigma})], \\ \rightarrow_{|\vec{\sigma}| \rightarrow \infty} z_W^\mu(\tau, \vec{\sigma}) &= Y^\mu(\tau) + \epsilon_r^\mu(\vec{h}) \sigma^r, \quad x^\mu(\tau) = z_F^\mu(\tau, 0), \\ g(\tau, 0) = g^r(\tau, 0) &= 0, \quad g(\tau, \vec{\sigma}) \rightarrow_{|\vec{\sigma}| \rightarrow \infty} 0, \quad g^r(\tau, \vec{\sigma}) \rightarrow_{|\vec{\sigma}| \rightarrow \infty} 0. \end{aligned} \quad (3.7)$$

In Ref.[3] there is the expression of the induced 4-metric $g_{FAB}(\tau, \vec{\sigma})$ (with $h_{Frs} = -\epsilon g_{Frs}$ being a positive-definite 3-metric), of the unit normal $l^\mu(\tau, \vec{\sigma})$ to the 3-space and of the associated lapse and shift functions.

To define the non-inertial rest-frame instant form we must find the form of the internal Poincare' generators. As shown in Ref.[3] Eq.(2.1) and the first of Eqs.(2.2) imply

$$\begin{aligned}
P^\mu &= M_C h^\mu = \int d^3\sigma \rho^\mu(\tau, \vec{\sigma}) \approx \\
&\approx h^\mu \int d^3\sigma \sqrt{\gamma(\tau, \vec{\sigma})} \left(\frac{\det(\delta_r^s + \partial_r g^s)}{\sqrt{\gamma_F}} T_{F\perp\perp} - \partial_r g h_F^{rs} T_{F\perp s} \right)(\tau, \vec{\sigma}) + \\
&+ \epsilon_u^\mu(\vec{h}) \int d^3\sigma \left(- \frac{\delta^{ua} \epsilon_{asr} \epsilon_{vwt} \partial_v g \partial_w g^s \partial_t g^r}{\sqrt{\gamma_F}} T_{F\perp\perp} - \right. \\
&- \left. (\delta_r^u + \partial_r g^u) h_F^{rs} T_{F\perp s} \right)(\tau, \vec{\sigma}) = \\
&\stackrel{def}{=} \int d^3\sigma \mathcal{T}_F^\mu(\tau, \vec{\sigma}), \tag{3.8}
\end{aligned}$$

so that the internal mass and the rest-frame conditions become (Eqs.(2.9) are recovered for the inertial rest frame)

$$\begin{aligned}
M_C &= \int d^3\sigma \left(\frac{\det(\delta_r^s + \partial_r g^s)}{\sqrt{\gamma}} T_{F\perp\perp} - \partial_r g h_F^{rs} T_{F\perp s} \right)(\tau, \vec{\sigma}), \\
\hat{\mathcal{P}}^u &= \int d^3\sigma \left(- \frac{\delta^{ua} \epsilon_{asr} \epsilon_{vwt} \partial_v g \partial_w g^s \partial_t g^r}{\sqrt{\gamma_F}} T_{F\perp\perp} - \right. \\
&- \left. (\delta_r^u + \partial_r g^u) h_F^{rs} T_{F\perp s} \right)(\tau, \vec{\sigma}) \approx 0. \tag{3.9}
\end{aligned}$$

Then it can be shown [3] that the second of Eqs.(2.2) together with Eqs. (2.8) and (2.6) imply the following form of the constraints eliminating the 3-center of mass and of the effective spin

$$\begin{aligned}
\hat{\mathcal{K}}^u &= \int d^3\sigma \left(g \left[\delta^{ur} \partial_r g T_{F\perp\perp} - (\delta_r^u + \partial_r g^u) h_F^{rs} T_{F\perp s} \right] - \right. \\
&- \left. (\sigma^u + g^u) \left[\frac{\det(\delta_r^s + \partial_r g^s)}{\sqrt{\gamma}} T_{F\perp\perp} - \partial_r g h_F^{rs} T_{F\perp s} \right] \right)(\tau, \vec{\sigma}) \approx 0, \\
\tilde{S}^r &\approx \hat{S}^r = \frac{1}{2} \delta^{rn} \epsilon_{nuv} \int d^3\sigma \left((\sigma^u + g^u) \left[\delta^{vm} \partial_m g T_{F\perp\perp} - (\delta_r^v + \partial_r g^v) h_F^{rs} T_{F\perp s} \right] - \right. \\
&- \left. (\sigma^v + g^v) \left[\delta^{um} \partial_m g T_{F\perp\perp} - (\delta_r^u + \partial_r g^u) h_F^{rs} T_{F\perp s} \right] \right)(\tau, \vec{\sigma}). \tag{3.10}
\end{aligned}$$

and these formulas allow to recover Eqs.(2.9) of the inertial rest frame.

Therefore the non-inertial rest-frame instant form of dynamics is well defined since we have: a) a decoupled center of mass carrying a pole-dipole structure; b) well defined internal Poincaré generators M_C , $\vec{\mathcal{P}} \approx 0$, $\vec{\tilde{S}}$, $\vec{\mathcal{K}} \approx 0$ at spatial infinity; c) non-Euclidean 3-spaces tending in a direction-independent way to space-like hyper-planes, where they are orthogonal to P^μ .

C. The Hamiltonian of the Non-Inertial Rest-Frame Instant Form

In Ref.[3] there is the determination of the effective Hamiltonian of the non-inertial rest-frame instant form replacing Mc of the inertial rest-frame one.

In conclusion the effective Hamiltonian $\mathcal{M}c$ of the non-inertial rest-frame instant form is not the internal mass Mc , since Mc describes the evolution from the point of view of the asymptotic inertial observers. There is an additional term interpretable as an inertial potential producing relativistic inertial effects (\mathcal{T}_F^μ is defined in Eq.(3.8))

$$\begin{aligned}\mathcal{M}c &= Mc + \int d^3\sigma \left(\frac{\partial g^r}{\partial \tau} \mathcal{T}_{F r} + \frac{\partial g}{\partial \tau} \mathcal{T}_{F \tau} \right)(\tau, \vec{\sigma}) = \\ &= \int d^3\sigma \epsilon \left(\left[h_\mu \left(1 + \frac{\partial g}{\partial \tau} \right) + \epsilon_{\mu r} \frac{\partial g^r}{\partial \tau} \right] \mathcal{T}_F^\mu \right)(\tau, \vec{\sigma}) = \\ &= \int d^3\sigma \sqrt{\gamma(\tau, \sigma^u)} \left((1 + n_F) T_{F \perp \perp} + n_F^r T_{F \perp r} \right)(\tau, \vec{\sigma})\end{aligned}\quad (3.11)$$

where

$$\begin{aligned}\sqrt{\gamma(\tau, \vec{\sigma})} T_{F \perp \perp}(\tau, \vec{\sigma}) &= \sum_i \delta^3(\sigma^u - \eta_i^u) \sqrt{m_i^2 c^2 + h_F^{rs}(\tau, \vec{\sigma}) \kappa_{ir}(\tau) \kappa_{is}(\tau)}, \\ \sqrt{\gamma(\tau, \vec{\sigma})} T_{F \perp r}(\tau, \vec{\sigma}) &= - \sum_i \delta^3(\sigma^u - \eta_i^u) \kappa_{ir}(\tau).\end{aligned}\quad (3.12)$$

D. The Non-Relativistic Limit of the Non-Inertial Rest Frame

In Ref.[11] there is the non-relativistic version of parametrized Minkowski theories. If x^a are the Cartesian coordinates of an inertial frame in Galilei space-time centered on an inertial observer, a global non-inertial frame can be defined by means of an invertible, global coordinate transformation (t is the absolute Newton time; $a = 1, 2, 3$; $\mathcal{A}^a(t, \vec{\sigma}) \rightarrow_{|\vec{\sigma}| \rightarrow \infty} A^a_b(t) \sigma^b$)

$$x^a = \mathcal{A}^a(t, \vec{\sigma}), \quad \text{with inverse} \quad \sigma^r = \mathcal{S}^r(t, \vec{x}). \quad (3.13)$$

The Jacobian of this transformation is

$$J^a_r(t, \vec{\sigma}) = \frac{\partial \mathcal{A}^a(t, \vec{\sigma})}{\partial \sigma^r}, \quad \det J(t, \vec{\sigma}) > 0, \quad (3.14)$$

and its inverse is denoted

$$\begin{aligned}\tilde{J}^r_a(t, \vec{\sigma}) &= \left[\frac{\partial \mathcal{S}^r(t, \vec{x})}{\partial x^a} \right]_{\vec{x}=\vec{\mathcal{S}}(t, \vec{\sigma})}, \\ J^a_r(t, \vec{\sigma}) \tilde{J}^r_b(t, \vec{\sigma}) &= \delta^a_b, \quad \tilde{J}^s_a(t, \vec{\sigma}) J^a_r(t, \vec{\sigma}) = \delta^s_r.\end{aligned}\quad (3.15)$$

The non-inertial frame is centered on an accelerated observer $\vec{x}_o(t) = \vec{\mathcal{A}}(t, 0)$. As shown in Section IV of Ref.[11] the traditional rigid non-inertial frames are contained in the definition (3.13) if we put $\mathcal{A}^a(t, \vec{\sigma}) = x_o^a(t) + \sigma^r R_{ra}(t)$ with $R(t)$ a time-dependent rotation matrix.

In Ref.[11] there is the definition of *parametrized Galilei theories* for isolated particle systems. The Lagrangian depends on the particle positions $\vec{\eta}_i(t)$ and on the functions $\vec{\mathcal{A}}(t, \vec{\sigma})$ as Lagrangian variables. Since the action is invariant under 3-diffeomorphisms the momenta $\vec{p}(t, \vec{\sigma})$, conjugate to the variables $\vec{\mathcal{A}}(t, \vec{\sigma})$, are determined by three first-class constraints (like Eqs.(2.1) of the relativistic case): $\rho^a(t, \vec{\sigma}) \approx \sum_i \delta^3(\sigma^r - \tilde{\eta}_i^r(t)) \tilde{J}^r_a(t, \vec{\sigma}) p_{ir}(t)$, where $\vec{p}_i(t)$ are the particle momenta. Therefore, the variables $\vec{\mathcal{A}}(t, \vec{\sigma})$ are gauge variables: a change of frame is a gauge transformation.

The resulting Galilei generators are

$$\begin{aligned}
E_{Galilei} &= H_c = \sum_{iars} \frac{1}{2m_i} \tilde{J}^r_a(t, \vec{\eta}_i(t)) p_{ir}(t) \tilde{J}^s_a(t, \vec{\eta}_i(t)) p_{is}(t), \\
P_{Galilei}^a &= \sum_{ir} \tilde{J}^r_a(t, \vec{\eta}_i(t)) p_{ir}(t), \\
J_{Galilei}^a &= \frac{1}{2} \sum_{bcd} \epsilon^{abc} \sum_i \left[\mathcal{A}^b(t, \vec{\eta}_i(t)) \delta^{cd} - \mathcal{A}^c(t, \vec{\eta}_i(t)) \delta^{bd} \right] \\
&\quad \tilde{J}^r_d(t, \vec{\eta}_i(t)) p_{ir}(t), \\
K_{Galilei}^a &= - \sum_i m_i \mathcal{A}^a(t, \vec{\eta}_i(t)),
\end{aligned} \tag{3.16}$$

where H_c is the canonical Hamiltonian. Instead the effective non-inertial Hamiltonian is

$$\mathcal{M}_{Galilei} = E_{Galilei} - \sum_i J^r_a(t, \vec{\eta}_i(t)) \frac{\partial \mathcal{A}^a(t, \vec{\eta}_i(t))}{\partial t} \Big|_{\vec{\eta}_i(t)} p_{ir}(t). \tag{3.17}$$

When one uses the functions $\mathcal{A}^a(t, \vec{\sigma}) = x_o^a(t) + \sigma^r R_{ra}(t)$, one recovers the standard Euler, Jacobi, Coriolis and centrifugal forces in the equation of motion of the particles (see Eq.(4.9) of Ref.[11]).

By using the Hamilton equations associated with the Hamiltonian (3.17), i.e. $\frac{d}{dt} B = \frac{\partial}{\partial t} B + \{B, \mathcal{M}_{Galilei}\}$, one can check that the Galilei generators (3.16) are constants of the motion.

Let us now consider the non-relativistic limit of the non-inertial rest frame, whose embedding is given in Eq.(3.7). We put $\tau = ct$ and $\vec{\eta}_i(\tau) = \vec{\eta}_i(t) = \vec{x}_{(n)i}(t)|_{\vec{x}_{(n)}(t)=\vec{p}_{(n)}=0}$, $\vec{\kappa}_i(\tau) = \vec{\kappa}_i(t) = \vec{p}_{(n)i}(t)|_{\vec{x}_{(n)}(t)=\vec{p}_{(n)}=0}$ and we use the notations of Subsection C of Section II.

The non-relativistic limit of the embedding (3.7) can be done by putting $\vec{h} = \frac{\vec{v}}{c} + O(c^{-2})$ and $\sigma^r + g^r(\tau, \vec{\sigma}) \rightarrow_{c \rightarrow \infty} \mathcal{A}^r(t, \vec{\sigma})$ and by assuming $g(\tau, \vec{\sigma}) = O(c^{-2})$. Then we get

$$\begin{aligned}
\frac{1}{c} z^o(\tau, \vec{\sigma}) &\rightarrow_{c \rightarrow \infty} t, \\
\vec{z}(\tau, \vec{\sigma}) &\rightarrow_{c \rightarrow \infty} \vec{y}_o + \vec{v}t + \vec{\mathcal{A}}(t, \vec{\sigma}).
\end{aligned} \tag{3.18}$$

If in Eq.(3.13) we put $\vec{x}_o(t) = \vec{\mathcal{A}}(t, \vec{0}) = 0$, we see that we are in a non-inertial frame centered on the Newton center of mass $\vec{x}_{(n)}(t) = \vec{y}_o + \vec{v}t$; if we put $\vec{p}_{(n)} = 0$ we are in a non-relativistic non-inertial rest frame.

The previous conditions imply the following expression for the induced 3-metric on the 3-space

$$\begin{aligned} h_{rs}(\tau, \vec{\sigma}) &= -\epsilon g_{rs}(\tau, \vec{\sigma}) = H_{rs}(t, \vec{\sigma}) + O(c^{-2}), \\ H_{rs} &= \delta_{rs} + \frac{\partial g^r}{\partial \sigma^s} + \frac{\partial g^s}{\partial \sigma^r} + \sum_u \frac{\partial g^u}{\partial \sigma^r} \frac{\partial g^u}{\partial \sigma^s}, \\ \sqrt{\gamma} &= \sqrt{H_{rs}} + O(c^{-2}). \end{aligned} \quad (3.19)$$

Then Eqs.(3.14), (3.15) and (3.19) imply (H^{rs} is the inverse of H_{rs})

$$\begin{aligned} J^a{}_r &= \delta_r^a + \frac{\partial g^a}{\partial \sigma^r}, \quad \text{with inverse } \tilde{J}^r{}_a, \\ H_{rs} &= \sum_a J^a{}_r J^a{}_s, \quad H^{rs} = \sum_a \tilde{J}^r{}_a \tilde{J}^s{}_a. \end{aligned} \quad (3.20)$$

With these notations the non-relativistic limit of the external and internal Poincaré generators (2.8), (3.9), (3.10) produces the following form for the Galilei generators (3.16)

$$\begin{aligned} P^0 &\rightarrow_{c \rightarrow \infty} mc + E_{Galilei}, \quad E_{Galilei} = H_c = \frac{\vec{p}_{(n)}^2}{2m} + \mathcal{E}_{Galilei}, \\ \vec{P} &= \vec{p}_{(n)} = \vec{P}_{Galilei}, \\ \vec{J} &= \vec{x}_{(n)}(t) \times \vec{p}_{(n)} + \vec{\mathcal{S}}_{Galilei} = \vec{J}_{Galilei}, \\ \frac{1}{c} \vec{K} &\rightarrow_{c \rightarrow \infty} t \vec{p}_{(n)} - m \vec{x}_{(n)} = \vec{K}_{Galilei}, \\ \mathcal{E}_{Galilei} &= \sum_i \frac{1}{2m_i} H^{rs}(t, \vec{\eta}_i(t)) \tilde{\kappa}_{ir}(t) \tilde{\kappa}_{is}(t), \\ \mathcal{S}_{Galilei}^r &= \sum_{uv} \epsilon^{ruv} \sum_i \mathcal{A}^u(t, \vec{\eta}_i(t)) J^v{}_r(t, \vec{\eta}_i(t)) \\ &\quad \sum_s H^{rs}(t, \vec{\eta}_i(t)) \tilde{\kappa}_{is}(t), \\ \mathcal{P}_{Galilei}^r &= \sum_{ius} J^r{}_u(t, \vec{\eta}_i(t)) H^{rs}(t, \vec{\eta}_i(t)) \tilde{\kappa}_{is}(t) \approx 0, \\ \mathcal{K}_{Galilei}^r &= - \sum_i m_i \mathcal{A}^r(t, \vec{\eta}_i(t)) \approx 0. \end{aligned} \quad (3.21)$$

The time constancy of the generators (3.16) implies the time constancy of the internal Galilei generators $\mathcal{E}_{Galilei}$, $\vec{\mathcal{S}}_{Galilei}$, $\vec{\mathcal{P}}_{Galilei}$, $\vec{\mathcal{K}}_{Galilei}$.

Let us remark that if we put $\vec{x}_o(t) = \vec{\mathcal{A}}(t, \vec{0}) \neq 0$, we are in an arbitrary non-inertial frame and we should study the non-relativistic limit of Eqs. (2.8), (3.3), (3.4).

IV. RELATIVISTIC N-BODY SYSTEMS

Let us remark that the Lagrangian, and therefore the energy-momentum tensor, of directly interacting N-particle systems is not known. Therefore we have to find by hand the ten generators of the internal Poincaré algebra in the inertial rest frame. For two-body systems this was done in Ref.[7] with $E_i(\tau) = \sqrt{m_i^2 c^2 + \vec{\kappa}_i^2(\tau) + \Phi(\vec{\rho}^2(\tau))}$ ($\vec{\rho} = \vec{\eta}_1 - \vec{\eta}_2$), $\epsilon p_i^2 = m_i^2 c^2 + \Phi(\vec{\rho}^2)$ and $\vec{K} = -\sum_i \vec{\eta}_i(\tau) E_i(\tau)$. See Ref.[7] for a review of the older attempts of solving the problem of a consistent relativistic mechanics.

In this Section we give a generalization of this two-body model to N particle systems. In this class of models the potentials in the Hamiltonian Mc appear under the square root as an addition to the rest mass in the particle kinetic energy. Instead, in absence of a Lagrangian, no closed form of the Lorentz boosts is known when the Hamiltonian has the form $Mc = \sum_i E_i + V$ ($E_i = \sqrt{m_i^2 c^2 + \vec{\kappa}_i^2}$).

However the Lagrangian is known for the two relevant cases of particles interacting with the electro-magnetic [4, 8, 9] and gravitational [17] fields. By solving the field equations with a no-incoming radiation condition we can find the energy-momentum tensor and the internal Poincaré generators for the N particles in these cases. These are the only cases in which we have an expression of the generators with the potentials appearing outside the square roots of the kinetic energy terms.

See Appendix A for the resulting Poincaré generators in the case of N charged scalar particles with Coulomb plus Darwin potential. Instead see Appendix B for such generators in the case of N scalar particles in the Hamiltonian Post-Minkowskian (HPM) linearization of tetrad gravity: the model can be seen as living in the inertial rest frame of the asymptotic Minkowski space-time with the extra inertial potential depending on the York time (it gives the trace of the extrinsic curvature of the HPM 3-spaces; the rest of the extrinsic curvature is dynamically determined). The terms in the ADM energy containing the HPM GW are the gravitational counterpart of the Darwin potential. This model replaces the non-relativistic one of Ref.[34] and tends to it in the non-relativistic case.

In this Section we give a simple example of N-body system, whose internal Poincaré generators have been found with a suitable canonical transformation starting from the free case.

If g is the coupling constant of the interaction, let us define a canonical transformation $U(g)$ ($U(0) = I$ in the free case) of the type ²¹

$$\begin{aligned} U(g) \vec{\eta}_i &= \vec{\eta}_i + \vec{U}_i(g; \vec{\eta}_l, \vec{\kappa}_l), \\ U(g) \vec{\kappa}_i &= \vec{\kappa}_i + \vec{W}_i(g; \vec{\eta}_l, \vec{\kappa}_l), \end{aligned} \tag{4.1}$$

such that the internal 3-momentum and angular momentum are left fixed

²¹ If $U(g) f(\vec{\eta}_k, \vec{\kappa}_k) = f(U(g) \vec{\eta}_k, U(g) \vec{\kappa}_k)$, then $\{U(g) f_1, U(g) f_2\} = U(g) \{f_1, f_2\}$.

$$U(g) \vec{\mathcal{P}} = \vec{\mathcal{P}}, \quad U(g) \vec{\mathcal{J}} = \vec{\mathcal{J}}. \quad (4.2)$$

The functions $\vec{U}_i(g; \vec{\eta}_l, \vec{\kappa}_l)$ and $\vec{W}_i(g; \vec{\eta}_l, \vec{\kappa}_l)$ are assumed to depend only upon relative positions $\vec{\eta}_l - \vec{\eta}_m$.

Then the other internal Poincaré generators (2.14) take the form

$$\begin{aligned} M(g) c &= U(g) M c = \sum_i \sqrt{m_i^2 c^2 + \vec{\kappa}_i^2 + V_i(g; \vec{\eta}_l - \vec{\eta}_m, \vec{\kappa}_k)}, \\ \mathcal{K}^r(g) &= U(g) \mathcal{K}^r = - \sum_i \left[\eta_i^r + U_i^r(g; \vec{\eta}_l - \vec{\eta}_m, \vec{\kappa}_k) \right] \sqrt{m_i^2 c^2 + \vec{\kappa}_i^2 + V_i(g; \vec{\eta}_l - \vec{\eta}_m, \vec{\kappa}_k)}, \\ V_i(g; \vec{\eta}_l - \vec{\eta}_m, \vec{\kappa}_k) &= 2 \vec{\kappa}_i \cdot \vec{W}_i(g; \vec{\eta}_l - \vec{\eta}_m, \vec{\kappa}_k) + \vec{W}_i^2(g; \vec{\eta}_l - \vec{\eta}_m, \vec{\kappa}_k). \end{aligned} \quad (4.3)$$

The simplest case is

$$\begin{aligned} U(g) \vec{\eta}_i &= \vec{\eta}_i, \\ U(g) \vec{\kappa}_i &= \vec{\kappa}_i + \frac{\partial \mathcal{F}}{\partial \vec{\eta}_i}(g; \vec{\eta}_l - \vec{\eta}_m), \end{aligned} \quad (4.4)$$

with $\mathcal{F}(g; \vec{\eta}_l - \vec{\eta}_m) = \bar{\mathcal{F}}(g; \vec{\rho}_a \cdot \vec{\rho}_b)$ by using Eqs.(2.18).

We get

$$\begin{aligned} M(g) c &= \sum_i E_i = \sum_i \sqrt{m_i^2 c^2 + \vec{\kappa}_i^2 + V_i(g; \vec{\eta}_l - \vec{\eta}_m, \vec{\kappa}_i)}, \\ \mathcal{K}^r(g) &= - \sum_i \eta_i^r E_i = - \sum_i \eta_i^r \sqrt{m_i^2 c^2 + \vec{\kappa}_i^2 + V_i(g; \vec{\eta}_l - \vec{\eta}_m, \vec{\kappa}_i)}, \\ V_i(g; \vec{\eta}_l - \vec{\eta}_m, \vec{\kappa}_i) &= 2 \vec{\kappa}_i \cdot \frac{\partial \mathcal{F}(g; \vec{\eta}_l - \vec{\eta}_m)}{\partial \vec{\eta}_i} + \left(\frac{\partial \mathcal{F}(g; \vec{\eta}_l - \vec{\eta}_m)}{\partial \vec{\eta}_i} \right)^2. \end{aligned} \quad (4.5)$$

Therefore we have $E_i = \sqrt{m_i^2 c^2 + \vec{\kappa}_i^2 + V_i(g; \vec{\eta}_l - \vec{\eta}_m, \vec{\kappa}_i)}$, i.e. $\epsilon p_i^2 = m_i^2 c^2 + V_i(g; \vec{\eta}_l - \vec{\eta}_m, \vec{\kappa}_i)$ like in the models with first-class constraints [7] and in the model of Ref.[27]. As it happens in the models of Appendices A and B generically the potentials leading to a correct Poincaré algebra are momentum dependent in the N-body case. Let us remark that the potentials V_i must be restricted in such a way that one has $\epsilon p_i^2 > 0$ for each massive particle. Therefore in this RCM the *inertial mass of the particles is modified by the interactions*.

For $\mathcal{F} = \frac{g}{2} \sum_a \vec{\rho}_a^2$, we get $\frac{\partial \mathcal{F}}{\partial \vec{\eta}_i} = g \sqrt{N} \sum_a \gamma_{ai} \vec{\rho}_a$ and $V_i = g \sqrt{N} \sum_a \gamma_{ai} \vec{\kappa}_i \cdot \vec{\rho}_a + g^2 N \sum_{ab} \gamma_{ai} \gamma_{bi} \vec{\rho}_a \cdot \vec{\rho}_b$.

The rest-frame conditions $\vec{\mathcal{P}} \approx 0$ and $\vec{\mathcal{K}} \approx 0$ and Eqs.(2.18) imply

$$\begin{aligned}
M(g)c &\approx \sum_i E_i = \sum_i \sqrt{m_i^2 c^2 + N \sum_{ab} \gamma_{ai} \gamma_{bi} \vec{\pi}_a \cdot \vec{\pi}_b + V_i(g; \vec{\rho}_b, \vec{\pi}_b)}, \\
\vec{\eta}_+ &\approx \vec{\eta}(\vec{\rho}_b, \vec{\pi}_b) = -\frac{1}{\sqrt{N}} \sum_{ia} \Gamma_{ai} \frac{\sqrt{m_i^2 c^2 + N \sum_{bc} \gamma_{bi} \gamma_{ci} \vec{\pi}_b \cdot \vec{\pi}_c + V_i(g; \vec{\rho}_b, \vec{\pi}_b)}}{\sum_k \sqrt{m_k^2 c^2 + N \sum_{bc} \gamma_{bk} \gamma_{ck} \vec{\pi}_b \cdot \vec{\pi}_c + V_k(g; \vec{\rho}_b, \vec{\pi}_b)}} \vec{\rho}_a.
\end{aligned} \tag{4.6}$$

Since the invariant mass Mc is the Hamiltonian for the τ -evolution in the Wigner 3-spaces, the total time derivative along the solutions of Hamilton equations, i.e. the Liouville operator, acting on particle "i" is ($V_i(\tau) = V_i(g; \vec{\eta}_l - \vec{\eta}_m, \vec{\kappa}_i)$)

$$\begin{aligned}
\hat{L}_i &= \frac{\partial}{\partial \tau} + \{\eta_i^r(\tau), Mc\} \frac{\partial}{\partial \eta_i^r} + \{\kappa_{ir}(\tau), Mc\} \frac{\partial}{\partial \kappa_{ir}} = \\
&= \frac{\partial}{\partial \tau} + \frac{1}{E_i(\tau)} \left(\kappa_{ir}(\tau) + \frac{1}{2} \frac{\partial V_i(\tau)}{\partial \kappa_{ir}} \right) \frac{\partial}{\partial \eta_i^r} - \sum_j \frac{1}{2 E_j(\tau)} \frac{\partial V_j(\tau)}{\partial \eta_i^r} \frac{\partial}{\partial \kappa_{ir}}.
\end{aligned} \tag{4.7}$$

When we take into account the rest-frame conditions $\vec{\mathcal{P}} \approx 0$ and $\vec{\mathcal{K}} \approx 0$, these operators have to be restricted to the Liouville operators for the relative variables $\vec{\rho}_a(\tau)$, $\vec{\pi}_a(\tau)$

$$\begin{aligned}
\hat{L}_a &= \frac{\partial}{\partial \tau} + \{\rho_a^r(\tau), Mc\} \frac{\partial}{\partial \rho_a^r} + \{\pi_a^r(\tau), Mc\} \frac{\partial}{\partial \pi_a^r} = \\
&= \frac{\partial}{\partial \tau} + \sum_i \frac{\gamma_{ai} \sum_b \gamma_{bi} \pi_b^r + \frac{1}{2} \frac{\partial V_i(g; \vec{\rho}_b, \vec{\pi}_b)}{\partial \pi_a^r}}{\sqrt{m_i^2 c^2 + N \sum_{bc} \gamma_{bi} \gamma_{ci} \vec{\pi}_b \cdot \vec{\pi}_c + V_i(g; \vec{\rho}_b, \vec{\pi}_b)}} \frac{\partial}{\partial \rho_a^r} - \\
&- \sum_i \frac{\frac{\partial V_i(g; \vec{\rho}_b, \vec{\pi}_b)}{\partial \rho_a^r}}{2 \sqrt{m_i^2 c^2 + N \sum_{bc} \gamma_{bi} \gamma_{ci} \vec{\pi}_b \cdot \vec{\pi}_c + V_i(g; \vec{\rho}_b, \vec{\pi}_b)}} \frac{\partial}{\partial \pi_a^r}.
\end{aligned} \tag{4.8}$$

Let us remark that collisions happen when $0 = \vec{\eta}_i(\tau) - \vec{\eta}_j(\tau) = \frac{1}{\sqrt{N}} \sum_a (\Gamma_{ai} - \Gamma_{aj}) \vec{\rho}_a(\tau)$.

The model can be reformulated in the non-inertial rest frames along the lines of Section III.

V. THE MICRO-CANONICAL ENSEMBLE

In this Section we will give a definition of the micro-canonical ensemble for an isolated system of N particles with arbitrary either short- or long-range interactions both in non-relativistic and relativistic classical mechanics and both in inertial and non-inertial rest frames. In this way we can get a description of the equilibrium configurations of the system both in non-relativistic and relativistic statistical mechanics not only in inertial rest frames but also in the non-inertial ones.

Firstly in Subsection A we recall the standard definition of the non-relativistic ordinary micro-canonical partition function $Z_{(nr,st)}(E, V, N)$ (see for instance ch.6 of Ref.[20]) and of its extended form $\tilde{Z}_{(nr,st)}(E, \vec{\mathcal{S}}, V, N)$ used in Ref.[30] ($Z_{(nr,st)}(E, V, N) = \int d^3\mathcal{S} \tilde{Z}_{(nr,st)}(E, \vec{\mathcal{S}}, V, N)$), depending not only on the volume V , on the particle number N and on the value E of the total conserved energy ($H = E$, where H is the non-relativistic Hamiltonian of the isolated system) but also on the value $\vec{\mathcal{S}}$ of the total conserved angular momentum \vec{J} .

Then in Subsection B both the ordinary and extended micro-canonical partition functions, $Z_{(nr)}(\mathcal{E}, V, N)$ and $\tilde{Z}_{(nr)}(\mathcal{E}, \vec{\mathcal{S}}, V, N)$, are defined in the *non-relativistic inertial rest frame* defined in Subsection D of Section II, where the center of mass of the isolated system is put at the origin of the 3-coordinates ($\vec{x}_{(nr)} = 0$) and the total energy $E = \frac{\vec{p}_{(nr)}^2}{2Nm} + \mathcal{E}$ is replaced with the internal energy \mathcal{E} , which is an invariant of the centrally extended Galilei algebra. These definitions use only the internal Galilei generators of the isolated system. It is possible to reintroduce the dependence on the center of mass and to recover the standard partition functions.

In Subsection C there is the main new result, namely the definition of the ordinary and extended micro-canonical partition functions, $Z(\mathcal{E}, V, N)$ and $\tilde{Z}(\mathcal{E}, \vec{\mathcal{S}}, V, N)$, in the *relativistic rest frame* of Section II. In Subsection D we study their transformation properties under Lorentz transformations. These definitions use only the internal Poincaré generators of the isolated system and do not depend on the decoupled external relativistic center of mass. Now \mathcal{E} is the conserved invariant mass Mc of the isolated system ($\vec{\mathcal{S}}$ is its rest spin) and it is not possible to reintroduce a dependence on the external center of mass as was possible in the non-relativistic case.

Both in the non-relativistic and relativistic cases we define the *micro-canonical temperature* (see Subsection E) and we show that it is a *Lorentz-scalar*. We do not discuss the canonical ensemble and the canonical temperature because we consider both short- and long-range interactions among the particles so that in general these ensembles are not equivalent to the micro-canonical ones.

Finally in Subsection F the two micro-canonical partition functions are defined in the *relativistic non-inertial rest frame* of Section III and it is shown how to make the non-relativistic limit to the *non-relativistic non-inertial rest frame* in Subsection G. These non-inertial partitions functions do not seem to have been defined till now. We introduce the problem of the notion of *non-inertial equilibrium* and of whether it could be gauge equivalent to *inertial equilibrium* at least in the passive viewpoint.

A. The Standard Micro-Canonical Ensemble in Non-Relativistic Inertial Frames

The standard non-relativistic micro-canonical distribution function $f_{(mc,nr,st)}(\vec{x}_1, \dots, \vec{p}_N | E, V, N)$ of a system of N particles with Hamiltonian H is defined in a spatial volume V ²² in the following way

$$\begin{aligned}
f_{(mc,nr,st)}(\vec{x}_1, \dots, \vec{p}_N | E, V, N) &= \frac{1}{Z_{(nr,st)}(E, V, N)} \chi(V) \delta(H_N(\vec{x}_1, \dots, \vec{p}_N) - E), \\
\chi(V) &= 1 \quad \text{for } \vec{x}_i \in V, \quad \chi(V) = 0 \quad \text{for } \vec{x}_i \notin V, \\
Z_{(nr,st)}(E, V, N) &= \int d\Gamma_N \chi(V) \delta(H_N(\vec{x}_1, \dots, \vec{p}_N) - E) = \frac{\partial \Omega_{(nr,st)}(E, V, N)}{\partial E}, \\
\Omega_{(nr,st)}(E, V, N) &= \int d\Gamma_N \chi(V) \theta(H_N(\vec{x}_1, \dots, \vec{p}_N) - E), \\
d\Gamma_N &= (N!)^{-1} \prod_{i=1, \dots, N} d^3 x_i d^3 p_i, \\
\int d\Gamma_N f_{(mc,nr,st)}(\vec{x}_1, \dots, \vec{p}_N | E, V, N) &= 1. \tag{5.1}
\end{aligned}$$

$Z_{(nr,st)}(E, V, N)$ is the standard micro-canonical partition function.

Due to the Hamilton equations of the particles, it satisfies $\frac{\partial f_{(mc,nr,st)}}{\partial t} + \{f_{(mc,nr,st)}, H\} = \hat{L} f_{(mc,nr,st)} = 0$ (Liouville theorem), where $\hat{L} = \frac{\partial}{\partial t} + \sum_i \left(\frac{\partial H}{\partial \vec{p}_i} \cdot \frac{\partial}{\partial \vec{x}_i} - \frac{\partial H}{\partial \vec{x}_i} \cdot \frac{\partial}{\partial \vec{p}_i} \right)$ is the Liouville operator. Since the system is isolated we have $\frac{\partial f_{(mc,nr,st)}}{\partial t} = 0$, so that we are in the framework of *equilibrium* statistical mechanics.

The statistical average of a function $F(\vec{x}, \vec{p}; \vec{x}_i, \vec{p}_i)$ in the micro-canonical ensemble is

$$F_{(mc)}(\vec{x}, \vec{p} | E, V, N) = \langle F \rangle_{(mc)} = \int d\Gamma_N F(\vec{x}, \vec{p}; \vec{x}_i, \vec{p}_i) f_{(mc,nr,st)}(\vec{x}_1, \dots, \vec{p}_N | E, V, N). \tag{5.2}$$

As shown in Eq.(C5) of Appendix C (compare with Ref.[61]) for N free particles of mass m ($H = \sum_{i=1}^N \frac{\vec{p}_i^2}{2m}$) we have the following expression for the micro-canonical distribution function (with a spherical volume $V = \frac{4}{3} \pi R^3$: $\chi(V) = 0$ for $\vec{x}_i^2 > R^2$ and $\chi(V) = 1$ for $\vec{x}_i^2 < R^2$)

$$Z_{(nr,st)}(E, V, N) = \frac{1}{N!} \frac{(\sqrt{2\pi m})^{3N} E^{3N/2} V^N}{E \Gamma(3N/2)} \theta(E). \tag{5.3}$$

²² There two points of view regarding the volume: a) it is non-dynamical (one considers only the motions of the isolated system of particles contained in it); b) it is dynamical (the particles have elastic reflections at the boundaries of the volume, so that they are not an isolated system). We consider only the case of isolated systems as it is done in Ref.[30].

The standard extended partition function $\tilde{Z}_{(nr,st)}(E, \vec{\mathcal{S}}, V, N)$ is used in Ref.[30], whose Hamiltonian is given in Eq.(B1) (long range Newtonian gravity interactions), without eliminating the center of mass. It is defined starting from the following extended distribution function

$$\tilde{f}_{(mc,nr,st)}(\vec{x}_1, \dots, \vec{p}_N | E, \vec{\mathcal{S}}, V, N) = \tilde{Z}_{(nr,st)}^{-1}(E, \vec{\mathcal{S}}, V, N) \chi(V) \delta(H_N(\vec{x}_1, \dots, \vec{p}_N) - E) \delta^3(\vec{\mathcal{S}}_N - \vec{\mathcal{S}}), \quad (5.4)$$

with $Z_{(nr,st)}(E, \vec{\mathcal{S}}, V, N) = \int d\Gamma_N \chi(V) \delta(H_N(\vec{x}_1, \dots, \vec{p}_N) - E) \delta^3(\vec{\mathcal{S}}_N - \vec{\mathcal{S}})$ (see Eq.(5.10) for its expression).

B. The Micro-Canonical Ensemble in the Non-Relativistic Inertial Rest Frame

Let us now reformulate the micro-canonical ordinary and extended partition functions in the non-relativistic rest frame of the N-particle system, in which the Newtonian center of mass is at rest and is chosen as the origin of the 3-coordinates of the Euclidean 3-spaces by using the generators of the Galilei group given in Eqs.(2.37). The rest frame conditions are $\vec{P}_{Galilei} = 0$ and $\vec{K}_{Galilei} = 0$ (implying $\vec{x}_{(n)} = 0$).

From Eqs (2.36) and (2.37) we get the following expression for the extended and ordinary micro-canonical partition functions for an isolated system of N particles ($-\frac{1}{m} \vec{K}_{Galilei,N} |_{\vec{P}_{Galilei,N}=0} = \vec{x}_{(n)} = 0$; the volume V is assumed to have the center in the origin; $\vec{\rho}_{(n)a}$ and $\vec{\pi}_{(n)a}$, $a = 1, \dots, N-1$ are relative variables)

$$\begin{aligned} \tilde{Z}_{(nr)}(\mathcal{E}, \vec{\mathcal{S}}, V, N) &= \frac{1}{N!} \int \prod_i^{1..N} d^3\eta_i \chi(V) \int \prod_j^{1..N} d^3\kappa_j \delta(E_{Galilei,N} - \mathcal{E}) \\ &\quad \delta^3(\vec{\mathcal{S}}_{Galilei,N} - \vec{\mathcal{S}}) \delta^3(\vec{\mathcal{P}}_{Galilei,N}) \delta^3\left(\frac{\vec{K}_{Galilei,N}}{m}\right) = \\ &= \frac{1}{N!} \int \prod_a^{1..N-1} d^3\rho_{(n)a} \chi(V) \int \prod_b^{1..N-1} d^3\pi_{(n)b} \\ &\quad \delta(E_{Galilei,N} - \mathcal{E}) \delta^3(\vec{\mathcal{S}}_{Galilei,N} - \vec{\mathcal{S}}), \\ Z_{(nr)}(\mathcal{E}, V, N) &= \int d^3\mathcal{S} Z_{(nr)}(\mathcal{E}, \vec{\mathcal{S}}, V, N) = \\ &= \frac{1}{N!} \int \prod_i^{1..N} d^3\eta_i \chi(V) \int \prod_j^{1..N} d^3\kappa_j \delta(E_{Galilei,N} - \mathcal{E}) \\ &\quad \delta^3(\vec{\mathcal{P}}_{Galilei,N}) \delta^3\left(\frac{\vec{K}_{Galilei,N}}{m}\right) = \\ &= \frac{1}{N!} \int \prod_a^{1..N-1} d^3\rho_{(n)a} \chi(V) \int \prod_b^{1..N-1} d^3\pi_{(n)b} \\ &\quad \delta(E_{Galilei,N} - \mathcal{E}) \end{aligned} \quad (5.5)$$

Since at the non-relativistic level it is possible to find a canonical basis of relative variables such that $E_{Galilei,N} = \frac{1}{2m} \sum_a^{1..N-1} \vec{\pi}_a^2 + \text{potentials}$, the ordinary micro-canonical distribution function $Z_{(nr)}(\mathcal{E}, V, N)$ is equal to the standard one with N-1 particles.

For the micro-canonical distribution function we have

$$\begin{aligned} f_{(mc,nr)}(\vec{\eta}_1, \dots, \vec{\kappa}_N | \mathcal{E}, V, N) &= Z_{(nr)}^{-1}(\mathcal{E}, V, N) \frac{\chi(V)}{N!} \delta(E_{Galilei,N} - \mathcal{E}) \delta^3(\vec{S}_{Galilei,N} - \vec{\mathcal{S}}) \\ &\quad \delta^3(\vec{\mathcal{P}}_{Galilei,N}) \delta^3\left(\frac{\vec{\mathcal{K}}_{Galilei,N}}{m}\right) \approx \\ &\approx \tilde{f}_{(mc,nr)}(\vec{\rho}_{(n)1}, \dots, \vec{\pi}_{(n)N-1} | \mathcal{E}, V, N). \end{aligned} \quad (5.6)$$

It satisfies the Liouville equation with Hamiltonian $H = \mathcal{E} = E_{Galilei}|_{\vec{P}_{(n)}=\vec{x}_{(n)}=0}$ (it corresponds to the Hamilton-Jacobi description of the center of mass in the rest frame centered on the center of mass). Moreover it satisfies $\partial_t f_{(mc,nr)} = 0$, so that it is an *equilibrium* distribution function in statistical mechanics. The statistical average of a function $F(\vec{x}, \vec{p}; \vec{\eta}_i, \vec{\kappa}_i)$ is

$$F_{(mc,nr)}(\vec{x}, \vec{p} | \mathcal{E}, V, N) = \int \prod_{i=1}^N d^3\eta_i d^3\kappa_i F(\vec{x}, \vec{p}; \vec{\eta}_i, \vec{\kappa}_i) f_{(mc,nr)}(\vec{\eta}_1, \dots, \vec{\kappa}_N | \mathcal{E}, V, N). \quad (5.7)$$

In the non-relativistic case, by using the results in Subsection D of Section II, we can undo the Hamilton-Jacobi transformation on the center of mass and we can recover Eq.(5.1) from the second of Eqs.(5.5)

$$Z_{(nr,st)}(E, V, N) = \int d^3x_{(nr)} d^3p_{(nr)} \theta(R - |\vec{x}_{(n)}|) Z_{(nr)}(E = \frac{\vec{p}_{(n)}^2}{2m} + \mathcal{E}, V, N). \quad (5.8)$$

This is possible because the Galilei energy generator is the sum of the kinetic energy of the center of mass and of the internal energy, which is an invariant at the non-relativistic level. This property does not exist at the relativistic level with the Poincaré group.

As shown in Appendix C (Eqs.(C17) and (C42)-(C43) with $\vec{\kappa}_+ = 0$) we get the following expressions for the standard and the extended micro-canonical distribution functions in the rest frame for N free particles of mass m

$$\begin{aligned} Z_{(nr)}(\mathcal{E}, V, N) &= \frac{1}{N!} \left(\sqrt{\frac{m}{2\pi}} \right)^{3N} \frac{\mathcal{E}^{(3N-5)/2}}{\Gamma((3N-3)/2)} \theta(\mathcal{E}) \\ &\quad \sqrt{\frac{32\pi}{N^3 m^9}} 3^{N-1} V^{N-1} \int_0^\infty x^2 dx \left(\frac{j_1(x)}{x} \right)^N, \end{aligned} \quad (5.9)$$

$$\begin{aligned}
Z_{(nr)}(\mathcal{E}, V, N, \vec{\mathcal{S}}) &= \\
&= \frac{1}{N!(2\pi)^{9N}} \left(\sqrt{8\pi^3}\right)^{N+1} \left(\sqrt{m^3}\right)^{N-1} \frac{(2\pi)^3}{(m)^3} \left(\frac{3V}{4\pi}\right)^{N-1} \\
&\times \prod_i^N \int_0^1 x_i^2 dx_i \int_0^\pi \sin \theta_i d\theta_i \int_0^{2\pi} d\phi_i \delta^3\left(\sum_{i=1}^N \vec{x}_i\right) \frac{(\mathcal{E} - \tilde{\mathcal{E}}(\vec{\eta}_i, \vec{\mathcal{S}}))^{(3N-3)/2-1}}{\Gamma((3N-3)/2)} \theta(\mathcal{E} - \tilde{\mathcal{E}}(\vec{\eta}_i, \vec{\mathcal{S}})),
\end{aligned} \tag{5.10}$$

in which

$$\begin{aligned}
\tilde{\mathcal{E}}(\vec{\eta}_i, \vec{\mathcal{S}}) &= \frac{(\mathcal{S}^1)^2}{2m \sum_{i=1}^N (\eta_i^1)^2} + \frac{(\mathcal{S}^2)^2}{2m \sum_{i=1}^N (\eta_i^2)^2} + \frac{(\mathcal{S}^3)^2}{2m \sum_{i=1}^N (\eta_i^3)^2} = \\
&= \frac{1}{2mR^2} \left(\frac{(\mathcal{S}^1)^2}{\sum_{i=1}^N x_i^2 \cos^2 \phi_i \sin^2 \theta_i} + \frac{(\mathcal{S}^2)^2}{\sum_{i=1}^N x_i^2 \sin^2 \phi_i \sin^2 \theta_i} + \frac{(\mathcal{S}^3)^2}{2m \sum_{i=1}^N x_i^2 \cos^2 \theta_i} \right).
\end{aligned} \tag{5.11}$$

C. The Micro-Canonical Ensemble in the Relativistic Inertial Rest Frame

Let us remark that in the relativistic case there exists the following definition of the standard micro-canonical partition function

$$Z_{(st)}(E, V, N) = \frac{1}{N!} \int \chi(V) \prod_I^{1..N} d^3x_i d^3p_i \delta(H_N - E), \tag{5.12}$$

in an arbitrary inertial frame in the free case with $H_N = \sum_{i=1}^{1..N} \sqrt{m^2 c^2 + \vec{p}_i^2}$. Its form is not known in closed form for $m \neq 0$ (for $m=0$ see Ref.[33]).

This definition is obtained by describing the N free particles in an inertial frame by means of their world-lines x_i^μ and of the conjugate momenta p_i^μ by putting by hand $x_1^o = \dots = x_N^o = x^o$ and by using the mass-shell conditions $\epsilon p_i^2 = m_i^2 c^2$ to eliminate the energies p_i^o 's. This description includes the center of mass but no consistent way to include interactions among the N particles is known. The inclusion of interactions was the motivation of the new relativistic classical and quantum mechanics of Ref.[1] in which the world-lines x_i^μ and their momenta p_i^μ are derived quantities as shown in Section II. In the new formulation the Wigner 3-vectors $\vec{\eta}_i(\tau)$ and $\vec{\kappa}_i(\tau)$ are the fundamental canonical variables together with the rest-frame conditions.

Instead in this Subsection we define the micro-canonical partition function (with given internal energy \mathcal{E} and given rest spin $\vec{\mathcal{S}}$ in the rest frame) inside the instantaneous Wigner 3-spaces of the rest frame (in the inertial frame centered on the Fokker-Pryce external 4-center of inertia with 3-velocity $\vec{h} = 0$) after the elimination of the internal 3-center of mass. We define everything in the inertial rest frame but without including the external center of mass \vec{z} , \vec{h} .

The new partition function will be defined in terms of the internal Poincaré generators living inside the Wigner 3-spaces Σ_τ . For N free particles they are given in Eq.(2.14), while for the simple model of N interacting particles defined in Section IV they are given in Eq.(4.5). See Appendix A and B for such generators in more realistic interacting cases.

The natural volume V is a spherical box centered on the Fokker-Pryce center of inertia in the Wigner 3-space ($|\vec{\eta}_i(\tau)| \leq R$). Let $\chi(V) = \prod_i \theta(R - |\vec{\eta}_i(\tau)|)$ be the characteristic function identifying the volume V . However, since the internal center of mass is eliminated, it is more convenient to use a characteristic function depending only on the relative variables, namely $\chi(V) = \prod_a \theta(2R - |\vec{\rho}_a|)$.

Then the extended and ordinary partition functions of the micro-canonical ensemble are ²³ (in what follows we have $M_N = M_N(\vec{\eta}_i, \vec{\kappa}_i)$; the Jacobian $J(\vec{\rho}_a, \vec{\pi}_a)$ is defined by $\delta^3(\frac{\vec{\mathcal{K}}_N}{M_N c}) = J(\vec{\rho}_a, \vec{\pi}_a) \delta^3(\vec{\eta} - \vec{\eta}_+(\vec{\rho}_a, \vec{\pi}_a))$)

$$\begin{aligned}
\tilde{Z}(\mathcal{E}, \vec{\mathcal{S}}, V, N) &= \frac{1}{N!} \int \prod_i^{1..N} d^3 \eta_i \chi(V) \int \prod_j^{1..N} d^3 \kappa_j \delta(M_N c^2 - \mathcal{E}) \\
&\quad \delta^3(\vec{S}_N - \vec{\mathcal{S}}) \delta^3(\vec{\mathcal{P}}_N) \delta^3\left(\frac{\vec{\mathcal{K}}_N}{M_N c}\right) = \\
&= \frac{1}{N!} \int d^3 \eta \prod_{a=1}^{N-1} d^3 \rho_a \chi(V) \int \prod_b^{1..N-1} d^3 \pi_b J(\vec{\rho}_a, \vec{\pi}_a) \delta^3(\vec{\eta} - \vec{\eta}_+(\vec{\rho}_a, \vec{\pi}_a)) \\
&\quad \delta(M_N(\vec{\rho}_a, \vec{\pi}_a) c^2 - \mathcal{E}) \delta^3\left(\sum_{a=1}^{N-1} \vec{\rho}_a \times \vec{\pi}_a - \vec{\mathcal{S}}\right), \\
Z(\mathcal{E}, V, N) &= \int d^3 \mathcal{S} \tilde{Z}(\mathcal{E}, \vec{\mathcal{S}}, V, N) = \\
&= \frac{1}{N!} \int \prod_i^{1..N} d^3 \eta_i \chi(V) \int \prod_j^{1..N} d^3 \kappa_j \delta(M_N c^2 - \mathcal{E}) \delta^3(\vec{\mathcal{P}}_N) \delta^3\left(\frac{\vec{\mathcal{K}}_N}{M_N c}\right) = \\
&= \frac{1}{N!} \int d^3 \eta \prod_{a=1}^{N-1} d^3 \rho_a \chi(V) \int \prod_b^{1..N-1} d^3 \pi_b J(\vec{\rho}_a, \vec{\pi}_a) \delta^3(\vec{\eta} - \vec{\eta}_+(\vec{\rho}_a, \vec{\pi}_a)) \\
&\quad \delta(M_N(\vec{\rho}_a, \vec{\pi}_a) c^2 - \mathcal{E}). \tag{5.13}
\end{aligned}$$

Their evaluation should be done with the methods introduced in Appendix C for the non-relativistic case, but the calculations are much more involved. In particular one should need a closed form for the inverse Laplace transform of multiple powers of modified Bessel functions.

Let us remark that the 3-vectors $\vec{\eta}_i(\tau)$, $\vec{\kappa}_i(\tau)$ are Wigner spin-1 3-vectors so that quantities like $\vec{\kappa}_i^2(\tau)$ are Lorentz scalars. The invariant mass Mc (and therefore \mathcal{E}) is a Lorentz

²³ When we have $\vec{\mathcal{K}} = -\sum_i \vec{\eta}_i E_i$, we get $-\vec{\mathcal{K}}/M_N c = \vec{\eta}_+ + \frac{1}{\sqrt{N}} \sum_a \frac{\sum_i \Gamma_{ai} E_i}{M_c} \vec{\rho}_a$ from Eqs.(2.18).

scalar. $\vec{\mathcal{P}}$ and $\vec{\mathcal{K}}/Mc$ are Wigner spin-1 3-vectors: under a Lorentz transformation Λ they undergo a Wigner rotation $R(\Lambda)$, so that expressions like $\delta^3(\vec{\mathcal{P}})$ are Lorentz scalars. Also the rest spin \vec{S} is a Wigner spin-1 3-vector. Therefore under a Lorentz transformation we get $\tilde{Z}(\mathcal{E}, \vec{\mathcal{S}}, V, N) \mapsto \tilde{Z}(\mathcal{E}, R(\Lambda)^{-1}\vec{\mathcal{S}}, V, N)$, (the volume is a Lorentz scalar because both $|\vec{\eta}_i(\tau)|$ and $|\vec{\rho}_a(\tau)|$ are Lorentz scalars). Instead $Z(\mathcal{E}, V, N)$ is a *Lorentz scalar*.

Now we have the distribution function

$$f_{(mc)}(\vec{\rho}_1, \dots, \vec{\pi}_{N-1} | \mathcal{E}, V, N) = Z^{-1}(\mathcal{E}, V, N) \frac{\chi(V)}{N!} \delta(M_N c^2 - \mathcal{E}) \delta^3(\vec{\mathcal{P}}_N) \delta^3(\frac{\vec{\mathcal{K}}_N}{M_N c}), \quad (5.14)$$

and its extended version

$$\begin{aligned} \tilde{f}_{(mc)}(\vec{\rho}_1, \dots, \vec{\pi}_{N-1} | \mathcal{E}, \vec{\mathcal{S}}, V, N) &= \tilde{Z}^{-1}(\mathcal{E}, \vec{\mathcal{S}}, V, N) \frac{\chi(V)}{N!} \delta(M_N c^2 - \mathcal{E}) \delta^3(\vec{S}_N - \vec{\mathcal{S}}) \\ &\delta^3(\vec{\mathcal{P}}_N) \delta^3(\frac{\vec{\mathcal{K}}_N}{M_N c}). \end{aligned} \quad (5.15)$$

It satisfies the Liouville theorem with $H = Mc$. Moreover it satisfies $\partial_\tau f_{(mc)} = 0$, so that it is an *equilibrium* distribution function in statistical mechanics. The statistical average of a function $F(\vec{\eta}, \vec{\kappa}; \vec{\eta}_i, \vec{\kappa}_i)$ is ($f_{(mc)}$ is considered as a function of $\vec{\eta}_i$ and $\vec{\kappa}_i$ through the equations defining the relative variables)

$$F_{(mc)}(\vec{\eta}, \vec{\kappa}) = \int \prod_{i=1}^N d^3\eta_i d^3\kappa_i F(\vec{\eta}, \vec{\kappa}; \vec{\eta}_i, \vec{\kappa}_i) f_{(mc)}(\vec{\rho}_1, \dots, \vec{\pi}_{N-1} | \mathcal{E}, V, N). \quad (5.16)$$

D. The Micro-Canonical Ensemble in the Relativistic Inertial Rest Frame as seen by an Arbitrary Inertial Observer

Eq.(5.13) is evaluated by considering the phase space measure over the Wigner 3-space Σ_τ associated with an arbitrary value of the proper time τ , where the $6N$ variables $\vec{\eta}_i(\tau)$, $\vec{\kappa}_i(\tau)$, restricted by the rest-frame conditions $\vec{\mathcal{P}}(\tau) = 0$ and $\vec{\mathcal{K}}(\tau) = 0$, depend on the independent $6N-6$ relative variables $\vec{\rho}_a(\tau)$, $\vec{\pi}_a(\tau)$. The invariant mass (the Hamiltonian) Mc and the rest spin \vec{S} depend on these relative variables, but are τ -independent being Casimir invariants (i.e. constants of the motion).

Unlike with the non-relativistic case we cannot reintroduce the external center of mass by using the frozen Jacobi data \vec{z} , \vec{h} : even if the measure $d^3z d^3h$ is Lorentz invariant, \vec{z} is a non-covariant quantity.

In the rest frame, where $\tau = x_{(cm)}^0 = \text{const.}$ in the Wigner 3-space Σ_τ , we can rewrite the distribution function in the following form

$$f_{(mc)}(\vec{\rho}_1(\tau), \dots, \vec{\pi}_{N-1}(\tau) | \mathcal{E}, V, N) = \tilde{f}_{(mc)}(\vec{\eta}_1(\tau), \dots, \vec{\kappa}_N(\tau) | \mathcal{E}, V, N), \quad (5.17)$$

due to Eqs.(2.18). Then Eqs.(2.27) allows us to rewrite it in the form

$$f_{(mc)}(\vec{\rho}_1(\tau), \dots, \vec{\rho}_{N-1}(\tau)|\mathcal{E}, V, N) = \hat{f}_{(mc)}(\vec{x}_{(cm)i}(x_{(cm)}^o) - \frac{\vec{z}_{(cm)}}{Mc}, \vec{p}_{(cm)i}(x_{(cm)}^o)|\mathcal{E}, V, N), \quad (5.18)$$

depending on the rest-frame world-lines and their momenta. Then Eqs.(2.35) allow to get the form

$$f_{(mc)}(\vec{\rho}_1(\tau), \dots, \vec{\rho}_{N-1}(\tau)|\mathcal{E}, V, N) = \check{f}_{(mc)}(\Lambda^k{}_\nu \tilde{x}_i^\nu(x^o) - \frac{\vec{z}_{(cm)}}{Mc}, \Lambda^k{}_\nu \tilde{p}_i^\nu(x^o)|\mathcal{E}, V, N), \quad (5.19)$$

in an arbitrary Lorentz frame in the 3-space Σ_{x^o} with $x^o = x_i^o(\tau_i) = \text{const.}$, see Eq. (2.30).

As a consequence, due to the frame-dependence of the Jacobi data $\vec{z}_{(cm)}$ of the external relativistic center of mass we cannot put $\vec{z}_{(cm)} = 0$ like in the non-relativistic case and get a micro-canonical distribution function of the type $F_{(mc)}(\vec{x}_i(x^o), \vec{p}_i(x^o)|\mathcal{E}, V, N)$, i.e. depending on the world-lines and their momenta in an arbitrary Lorentz frame.

However the basic obstruction to get this type of distribution function is that the Poincaré energy cannot be written as the center-of-mass energy plus an internal energy like in the case of the Galilei group.

E. The Micro-Canonical Temperature in the Non-Relativistic and Relativistic Inertial Rest-Frame

In the standard non-relativistic micro-canonical ensemble the *micro-canonical entropy* is

$$S_{(mc,nr,st)}(E, V, N) = \frac{1}{N} \ln Z_{(nr,st)}(E, V, N). \quad (5.20)$$

and the *micro-canonical temperature* $T_{(mc)} = T_{(mc)}(E, V, N)$ (see Refs.[33–35]; k_B is the Boltzmann constant) is

$$\frac{1}{k_B T_{(mc)}} = \frac{\partial S_{(mc,nr,st)}(E, V, N)}{\partial E} \Big|_{V,N} = \frac{1}{Z_{(nr,st)}(E, V, N)} \frac{\partial Z_{(nr,st)}(E, V, N)}{\partial E} \Big|_{V,N}. \quad (5.21)$$

Then one can introduce the Gibb's relation $dE = T_{(mc)} dS_{(mc,nr,st)} - P_{(mc)} dV + \mu_{(mc)} dN$ ($\mu_{(mc)}$ chemical potential) with $\frac{P_{(mc)}}{k_B T_{(mc)}} = \frac{\partial S_{(mc)}(E, V, N)}{\partial V} \Big|_{E,N}$ and $\frac{\mu_{(mc)}}{k_B T_{(mc)}} = \frac{\partial S_{(mc)}(E, V, N)}{\partial N} \Big|_{E,V}$, and the second law of thermodynamics $dS_{(mc,nr,st)} \geq 0$. With long range forces the micro-canonical ensemble is inequivalent to the canonical ensemble (in which there is negative heat capacity) as shown Refs. [31].

These definitions can be adapted to the non-relativistic rest frame and then extended to the relativistic rest frame by replacing the micro-canonical entropy (5.20) with the entropies $S_{(mc,nr)}(\mathcal{E}, V, N) = \frac{1}{N} \ln Z_{(nr)}(\mathcal{E}, V, N)$ and $S_{(mc)}(\mathcal{E}, V, N) = \frac{1}{N} \ln Z(\mathcal{E}, V, N)$, respectively.

This implies that in the relativistic inertial rest frame the micro-canonical temperature $T_{(mc)}$ ($\frac{1}{k_B T_{(mc)}} = \frac{1}{Z(\mathcal{E}, V, N)} \frac{\partial Z(\mathcal{E}, V, N)}{\partial \mathcal{E}}|_{V, N}$) is a *Lorentz scalar*, because the relativistic internal energy \mathcal{E} is a Lorentz scalar like the internal energy Mc^2 .

Therefore in the short range case (equivalence of the micro-canonical and canonical ensembles) the thermodynamic limit $N, V \rightarrow \infty$ with $N/V = \text{const.}$ gives rise to a canonical temperature T , limit of $T_{(mc)}(E, V, N)$, which is a *Lorentz scalar*. Therefore in the relativistic rest-frame instant form of dynamics we have $T = T_{rest}$ (see the Introduction for the existing three points of view).

In the case of the ideal Boltzmann gas (N free non-relativistic particles of mass m and energy $\frac{p^2}{2m}$) Eqs. (5.21) and (5.3) imply $\frac{1}{k_B T_{(mc)}} \rightarrow_{N \rightarrow \infty} \frac{3N}{2E}$ (like in the classical virial theorem). Moreover one gets $p_{(mc)} = k_B T_{(mc)} \frac{\partial S_{(mc, nr, st)}(E, V, N)}{\partial V}|_{E, N \rightarrow N \rightarrow \infty} k_B T_{(mc)} \frac{N}{V}$ and the resulting equation of state is $p_{(mc)} V = N k_B T_{(mc)}$. When the thermodynamics limit is well defined, then $T_{(mc)}$ and $p_{(mc)}$ become the canonical temperature and pressure, respectively.

These results can be reproduced also in the non-relativistic inertial rest frame by replacing Eq.(5.3) with Eq.(5.9) as shown in Subsection 5 of Appendix C.

Instead in the relativistic rest frame we are not able to find an explicit analytic form of the micro-canonical entropy $S_{(mc)}(\mathcal{E}, V, N) = \frac{1}{N} \ln Z(\mathcal{E}, V, N)$ (see Subsection 6 of Appendix C), so that we cannot obtain an explicit equation of state for a relativistic ideal Boltzmann gas (N free relativistic particles of mass m and energy $\sqrt{m^2 c^2 + \vec{k}^2}$). However in Ref.[21], by using the equilibrium Jüttner one particle distribution function and Eqs.(7.1) - (7.2) (see the next two Sections), it is shown that also in the relativistic case one obtains $p = \frac{N}{V} k_B T$.

F. The Micro-Canonical Ensemble in the Relativistic Non-Inertial Rest Frame

Let us extend the definition (5.13) of the extended micro-canonical partition function to the relativistic non-inertial rest frames by using the 10 asymptotic Poincaré generators given in Eqs.(3.9) and (3.10) (see also Ref.[3] Section 5.3 (p.77)) and the non-inertial Hamiltonian $\mathcal{M}c$ of Eq.(3.11). The non-inertial extended partition function, depending on the inertial potentials g and g^r appearing in Eq.(3.7), is

$$\begin{aligned} \tilde{Z}(\mathcal{E}, \vec{\mathcal{S}}, V, N | g, g^r) &= \frac{1}{N!} \int \chi(V) \prod_{i=1}^N d^3 \eta_i d^3 \kappa_i \delta(\mathcal{M}_N(\vec{\eta}_i, \vec{\kappa}_i; g, g^r) c^2 - \mathcal{E}) \\ &\quad \delta^3(J_N^u(\vec{\eta}_i, \vec{\kappa}_i; g, g^r) - \mathcal{S}^u) \delta^3(\hat{\mathcal{P}}_N^u(\vec{\eta}_i, \vec{\kappa}_i; g, g^r)) \delta^3\left(\frac{\hat{\mathcal{K}}_N^u(\vec{\eta}_i, \vec{\kappa}_i; g, g^r)}{\mathcal{M}_N(\vec{\eta}_i, \vec{\kappa}_i; g, g^r)c}\right), \end{aligned} \quad (5.22)$$

Let us remark that now we cannot introduce relative variables, because they are not tensorially defined in non-Euclidean 3-spaces. In the interacting case with potentials $V(\vec{\eta}_i(\tau) - \vec{\eta}_j(\tau); \vec{\kappa}_i(\tau)) = \tilde{V}(\sqrt{(\vec{\eta}_i(\tau) - \vec{\eta}_j(\tau))^2}; \vec{\kappa}_i(\tau))$ we must replace the quantity $(\vec{\eta}_i(\tau) - \vec{\eta}_j(\tau))^2$ with the bi-scalar Synge world function for Riemannian 3-spaces $\sigma_{(ij)}(\vec{\eta}_i(\tau), \vec{\eta}_j(\tau)) = \frac{1}{2}(\lambda_1 - \lambda_0) \int_{\lambda_0}^{\lambda_1} d\lambda {}^3g_{rs}(\eta_{(ij)}(\lambda, \tau)) \frac{\partial \eta_{(ij)}^r(\lambda, \tau)}{\partial \lambda} \frac{\partial \eta_{(ij)}^s(\lambda, \tau)}{\partial \lambda}$, where $\eta_{(ij)}(\lambda, \tau)$ is

the 3-geodesic joining $\vec{\eta}_i(\tau)$ and $\vec{\eta}_j(\tau)$ (see Ref. [33]). The momenta are covectors defined at the positions of the particles

In this case the distribution function is

$$\begin{aligned} \tilde{f}_{(mc)}(\vec{\rho}_1, \dots, \vec{\rho}_{N-1} | \mathcal{E}, \vec{\mathcal{S}}, V, N | g, g^r) &= \tilde{Z}^{-1}(\mathcal{E}, \vec{\mathcal{S}}, V, N | g, g^r) \chi(V) \delta(M_N(\vec{\eta}_i, \vec{\kappa}_i; g, g^r) c^2 - \mathcal{E}) \\ &\delta^3(J_N^u(\vec{\eta}_i, \vec{\kappa}_i; g, g^r) - \mathcal{S}^u) \delta^3(\hat{\mathcal{P}}_N^u(\vec{\eta}_i, \vec{\kappa}_i; g, g^r)) \delta^3\left(\frac{\hat{\mathcal{K}}_N^u(\vec{\eta}_i, \vec{\kappa}_i; g, g^r)}{M_N(\vec{\eta}_i, \vec{\kappa}_i; g, g^r)c}\right). \end{aligned} \quad (5.23)$$

It satisfies the Liouville theorem with the Hamiltonian \mathcal{M} of Eq.(3.11). In the non-inertial rest frames $J_N^u(\vec{\eta}_i, \vec{\kappa}_i; g, g^r)$, $\hat{\mathcal{P}}_N^u(\vec{\eta}_i, \vec{\kappa}_i; g, g^r)$, $\frac{\hat{\mathcal{K}}_N^u(\vec{\eta}_i, \vec{\kappa}_i; g, g^r)}{M_N(\vec{\eta}_i, \vec{\kappa}_i; g, g^r)c}$ are asymptotic constants of the motion at spatial infinity. As a consequence we have $\partial_\tau f_{(mc)} = 0$ notwithstanding the presence of the time-dependent long-range inertial potentials. Therefore in this passive viewpoint (we do not actively accelerate the gas but we go passively from an inertial to a non-inertial frame) we get an equilibrium distribution function also in non-inertial rest frames in accord with their gauge equivalence to the inertial ones shown in Section II.

By using a definition of entropy like in Eq.(5.20) the micro-canonical temperature turns out to be a functional of the inertial potentials.

G. The Micro-Canonical Ensemble in the Non-Relativistic Non-Inertial Rest Frame

By using the results of Subsection D of Section III for the form (3.21) of the non-inertial Galilei generators we get the following definition of the non-relativistic extended partition function in non-inertial rest frames

$$\begin{aligned} \tilde{Z}_{(nr)}(\mathcal{E}, \vec{\mathcal{S}}, V | g, g^r) &= \frac{1}{N!} \int \prod_i^{1..N} d^3\eta_i \chi(V) \int \prod_j^{1..N} d^3\kappa_j \delta(E_{Galilei,N} - \mathcal{E}) \\ &\delta^3(\vec{S}_{Galilei,N} - \vec{\mathcal{S}}) \delta^3(\vec{\mathcal{P}}_{Galilei,N}) \delta^3\left(\frac{\vec{\mathcal{K}}_{Galilei,N}}{m}\right). \end{aligned} \quad (5.24)$$

For the distribution function we have

$$\begin{aligned} \tilde{f}_{(mc,nr)}(\vec{\rho}_1, \dots, \vec{\rho}_{N-1} | \mathcal{E}, \vec{\mathcal{S}}, V, N | g, g^r) &= \tilde{Z}_{(nr)}^{-1}(\mathcal{E}, \vec{\mathcal{S}}, V, N | g, g^r) \chi(V) \delta(E_{Galilei,N} - \mathcal{E}) \\ &\delta^3(\vec{S}_{Galilei,N} - \vec{\mathcal{S}}) \delta^3(\vec{\mathcal{P}}_{Galilei,N}) \delta^3\left(\frac{\vec{\mathcal{K}}_{Galilei,N}}{m}\right). \end{aligned} \quad (5.25)$$

By using the Hamilton equations generated by the Hamiltonian (3.17) it can be checked that it satisfies the Liouville theorem. Due to the asymptotic constancy of the Galilei generators at spatial infinity we get again an equilibrium distribution function, $\partial_t \tilde{f}_{(mc,nr)} = 0$, like in inertial frames.

VI. ON THE ONE-PARTICLE DISTRIBUTION FUNCTION AND ON THE BOLTZMANN EQUATION IN RELATIVISTIC KINETIC THEORY IN THE INERTIAL REST FRAME

As shown in Ref.[20] in non-relativistic kinetic theory of diluted gases one can introduce the (non-equilibrium) one-particle distribution function

$$\begin{aligned}
 f(\vec{x}, \vec{p}, t) &= \langle \sum_i \delta^3(\vec{x} - \vec{x}_i) \delta^3(\vec{p} - \vec{p}_i) \rangle_{(Gibbs)} = \\
 &= \int \prod_{i=1}^N d^3x_i d^3p_i \sum_i \delta^3(\vec{x} - \vec{x}_i) \delta^3(\vec{p} - \vec{p}_i) \rho(\vec{x}_1, \vec{p}_1, \dots, \vec{x}_N, \vec{p}_N, t) = \\
 &= N \int \prod_{i=2}^N d^3x_i d^3p_i \rho(\vec{x}_1, \vec{p}_1, \dots, \vec{x}_N, \vec{p}_N, t),
 \end{aligned} \tag{6.1}$$

by means of an average on a Gibbs ensemble. The normalized density function $\rho(\vec{x}_1, \vec{p}_1, \dots, \vec{x}_N, \vec{p}_N, t)$ ($\int \prod_{i=1}^N d^3x_i d^3p_i \rho(\vec{x}_1, \vec{p}_1, \dots, \vec{x}_N, \vec{p}_N, t) = 1$) is symmetric in the exchange of particles (all equal with mass m) and satisfies the Liouville theorem ($\frac{d\rho}{dt} = \frac{\partial \rho}{\partial t} + \sum_{i=1}^N (\frac{\partial \rho}{\partial \vec{x}_i} \cdot \frac{d\vec{x}_i}{dt} + \frac{\partial \rho}{\partial \vec{p}_i} \cdot \frac{d\vec{p}_i}{dt}) = 0$) implied by the Hamilton equations for the N particles with given Hamiltonian H .

The function $f(\vec{x}, \vec{p}, t)$ satisfies the Boltzmann transport equation (\vec{F} is an external force)

$$\left(\frac{\partial}{\partial t} + \frac{\vec{p}}{m} \cdot \frac{\partial}{\partial \vec{x}} + \vec{F} \cdot \frac{\partial}{\partial \vec{p}} \right) f(\vec{x}, \vec{p}, t) = \left(\frac{\partial f}{\partial t} \right)_{coll}, \tag{6.2}$$

where $\left(\frac{\partial f}{\partial t} \right)_{coll}$ is the collision term ²⁴. In the case of free particles this term is an integral whose argument is a bilinear in the distribution function and linear in the differential cross section of the elastic scattering of pairs of particles.

In absence of external forces an equilibrium distribution is assumed to be independent from \vec{x} and to satisfy $\frac{\partial}{\partial t} f_{(eq)}(\vec{x}, \vec{p}, t) = 0$. Therefore it is a solution of the Boltzmann equation $\left(\frac{\partial f_{(eq)}}{\partial t} \right)_{coll} = 0$. In the case of free particles the solution is the Maxwell-Boltzmann distribution function $e^{-\beta \frac{\vec{p}^2}{2m}}$ with $\beta = 1/k_B T$ (T the canonical temperature).

The Boltzmann equation for $f(\vec{x}, \vec{p}, t)$ can be derived as an approximation starting from the coupled equations of motion (the BBGKY hierarchy) for the s -particle distribution functions $f_s(\vec{x}_1, \vec{p}_1, \dots, \vec{x}_s, \vec{p}_s, t) = \frac{N!}{(N-s)!} \int \prod_{i=s+1}^N d^3x_i d^3p_i \rho(\vec{x}_1, \vec{p}_1, \dots, \vec{x}_N, \vec{p}_N, t)$ by using the Liouville theorem.

Till now, in absence of a consistent RCM for N interacting particles, the relativistic Boltzmann equation is either postulated or derived from the Klein-Gordon quantum field theory (see Refs.[46], [21], [47]). In absence of external forces and assuming $p^o = \sqrt{m^2 c^2 + \vec{p}^2}$

²⁴ The hypothesis of molecular chaos is needed for its evaluation. Moreover the distribution function must change slowly over distances and times of the order of the characteristic interaction lengths and durations.

it takes the form $p_\mu \frac{\partial}{\partial x_\mu} f(x^\alpha, p^\alpha) = C(x^\alpha, p^\alpha)$ with the second member being the collision term (it has the same structure as in the non-relativistic case, but now it depends on the relativistic differential cross section for elastic scattering).

For free particles the equilibrium solution of the standard relativistic Boltzmann equation [21], [47], is the equilibrium homogeneous Boltzmann-Jüttner distribution [48]²⁵

$$f_{(eq)}(\vec{x}, \vec{p}) \underset{\text{rest frame}}{=} f_{(eq)}(\vec{p}) = A e^{-\beta^\mu p_\mu} = A e^{-\beta \sqrt{m^2 + \vec{p}^2}}, \quad (6.3)$$

where T is the canonical temperature in the rest frame (the 4-vector $\beta^\mu = \frac{1}{k_B T} U^\mu$, with U^μ a unit 4-vector equal to $(1; 0)$ in the rest frame, is a Killing vector; the limit $N, V \rightarrow \infty$ with $N/V = \text{const.}$ is assumed) and $A = \frac{\text{const.}}{4\pi k_B T m^2 c K_2(\frac{mc^2}{k_B T})}$. See for instance the first paper in

Ref.[44]. The non-relativistic limit reproduces the Maxwell-Boltzmann distribution $e^{-\beta \frac{\vec{p}^2}{2m}}$.

An open problem is whether $f(\vec{x}, \vec{p}, t)$ is a Lorentz scalar: this is discussed in Ref. [44], where there is also a discussion of the existing alternatives to the Jüttner distribution and their rebuttal.

In Subsection A we delineate how to define the relativistic one-particle distribution function $f(\vec{\eta}, \vec{\kappa}, \tau)$ in the Wigner 3-spaces of the inertial rest frame, showing that in our approach it is a *Lorentz scalar*.

In the case of free particles this scalar can be postulated to be solution of a relativistic Boltzmann equation of the type $\left(\sqrt{m^2 c^2 + \vec{\kappa}^2} \frac{\partial}{\partial \tau} + \vec{\kappa} \cdot \frac{\partial}{\partial \vec{\eta}}\right) f(\vec{\eta}, \vec{\kappa}, \tau) = C(\vec{\eta}, \vec{\kappa}, \tau)$ with the collision term evaluated in the standard way. However in this evaluation inside the Wigner 3-spaces of the inertial rest frame the elastic scattering of the two particles is described in their rest frame and their 3-coordinates $\vec{\eta}_1$ and $\vec{\eta}_2$ are kinematically restricted by the rest-frame condition $\vec{\mathcal{K}} = \sum_{i=1}^2 \vec{\eta}_i \sqrt{m^2 c^2 + \vec{\kappa}_i^2} = 0$. These conditions are compatible with the invariant differential cross section appearing in the collision term.

With this ansatz on the relativistic Boltzmann equation for free particles the equilibrium one-particle distribution function is the Jüttner distribution $f_{(eq)}(\vec{\eta}, \vec{\kappa}) = f_{(eq)}(\vec{\kappa}) = A e^{-\frac{\sqrt{m^2 c^2 + \vec{\kappa}^2}}{k_B T}}$. In our approach one expects that the 4-vector β^μ has the form $\beta^\mu = \frac{1}{k_B T} h^\mu$ with $h^\mu = P^\mu / \sqrt{\epsilon P^2}$ and with T being the canonical temperature.

Then in Subsection B we delineate which are the problems in trying to justify this relativistic Boltzmann equation when one starts from a relativistic BBGKY hierarchy based on a model like the one in Section IV for N interacting relativistic particles.

²⁵ It satisfies $p_\mu \frac{\partial}{\partial x_\mu} f_{(eq)}(x^\alpha, p^\alpha) = 0$ and $C(x^\alpha, p^\alpha) = 0$; the \vec{x} -dependence is absent if there are no global rigid rotations.

A. The One-Particle Distribution Function in the Relativistic Inertial Rest Frame

Let us consider a particle in the Wigner 3-space of the inertial rest frame with canonical coordinates $\vec{\eta}(\tau)$, $\vec{\kappa}(\tau)$, whose world-line is $x^\mu(\tau) = Y^\mu(\tau) + \epsilon_r^\mu(\vec{h}) \eta^r(\tau)$ according to Eq.(2.11).

A relativistic one-particle distribution function in the inertial rest frame can be defined by considering a normalized density function for a relativistic Gibbs ensemble of the type $\rho(\vec{\eta}_1, \vec{\kappa}_1, \dots, \vec{\eta}_N, \vec{\kappa}_N, \tau)$. Then the natural definition for the one-particle distribution function in the Wigner 3-spaces without a dependence on both the external and internal centers of mass is

$$\begin{aligned} f(\vec{\eta}(\tau), \vec{\kappa}(\tau), \tau) &= \langle \sum_i \delta^3(\vec{\eta}(\tau) - \vec{\eta}_i(\tau)) \delta^3(\vec{\kappa}(\tau) - \vec{\kappa}_i(\tau)) \rangle_{(Gibbs)} = \\ &= \int \prod_i^{1..N} d^3\eta_i d^3\kappa_i \rho(\vec{\eta}_1, \vec{\kappa}_1, \dots, \vec{\eta}_N, \vec{\kappa}_N, \tau) \delta^3(\vec{\mathcal{P}}_N) \delta^3\left(\frac{\vec{\mathcal{K}}_N}{M_N c}\right) \\ &\quad \left[\sum_i \delta^3(\vec{\eta}(\tau) - \vec{\eta}_i(\tau)) \delta^3(\vec{\kappa}(\tau) - \vec{\kappa}_i(\tau)) \right], \end{aligned} \quad (6.4)$$

with the following normalization of the density function, $\int \prod_i^{1..N} d^3\eta_i d^3\kappa_i \rho(\vec{\eta}_1, \vec{\kappa}_1, \dots, \vec{\eta}_N, \vec{\kappa}_N, \tau) \delta^3(\vec{\mathcal{P}}_N) \delta^3\left(\frac{\vec{\mathcal{K}}_N}{M_N c}\right) = 1$.

In the case of an equilibrium ensemble the replacement $\rho(\vec{\eta}_1, \vec{\kappa}_1, \dots, \vec{\eta}_N, \vec{\kappa}_N, \tau) \mapsto Z^{-1}(\mathcal{E}, V, N) \frac{\chi(V)}{N!} \delta(M_N c^2 - \mathcal{E})$ transforms Eq.(6.4) in a statistical average in the relativistic micro-canonical ensemble of the type of Eq.(5.16). Therefore the density function depends only on relative variables, $\rho(\vec{\eta}_1, \vec{\kappa}_1, \dots, \vec{\eta}_N, \vec{\kappa}_N, \tau) \delta^3(\vec{\mathcal{P}}_N) \delta^3\left(\frac{\vec{\mathcal{K}}_N}{M_N c}\right) = \tilde{\rho}(\vec{\rho}_1, \vec{\pi}_1, \dots, \vec{\rho}_{N-1}, \vec{\pi}_{N-1}, \tau) \delta^3(\vec{\mathcal{P}}_N) \delta^3\left(\frac{\vec{\mathcal{K}}_N}{M_N c}\right)$, and has the same transformation properties like the micro-canonical distribution function $f_{(mc)}(\vec{\rho}_1, \dots, \vec{\pi}_{N-1} | \mathcal{E}, V, N)$ of Eq.(5.14).

The use of Eq.(2.18) allows us to rewrite Eq.(6.4) in the following form

$$\begin{aligned} f(\vec{\eta}(\tau), \vec{\kappa}(\tau), \tau) &= \int \prod_a^{1..N-1s} d^3\rho_a d^3\pi_a \tilde{\rho}(\vec{\rho}_1, \vec{\pi}_1, \dots, \vec{\rho}_{N-1}, \vec{\pi}_{N-1}, \tau) \\ &\quad \sum_{i=1}^N \left(\delta^3(\vec{\kappa}(\tau) - \sqrt{N} \sum_{a=1}^{N-1} \gamma_{ai} \vec{\pi}_a) \right. \\ &\quad \left. \delta^3(\vec{\eta}(\tau) - \vec{\eta}_+ (\vec{\rho}_a, \vec{\pi}_a) - \frac{1}{\sqrt{N}} \sum_{a=1}^{N-1} \Gamma_{ai} \vec{\rho}_a) \right), \\ &\quad \int \prod_a^{1..N-1} d^3\rho_a d^3\pi_a \tilde{\rho}(\vec{\rho}_1, \vec{\pi}_1, \dots, \vec{\rho}_{N-1}, \vec{\pi}_{N-1}, \tau) = 1. \end{aligned} \quad (6.5)$$

By using Eq.(2.27) we can rewrite the internal Poincaré generators in terms of $\vec{x}_{(cm)i}(\tau = x_{(cm)}^0)$ and $\vec{p}_{(cm)i}(\tau = x_{(cm)}^0)$ and we have $d^3\eta_i d^3\kappa_i = d^3x_{(cm)i} d^3p_{(cm)i}$. Since the density

function depends only on the relative variables, the dependence on the non-covariant Jacobi data $\vec{z}_{(cm)}$ of the external center of mass drops out. Therefore Eq.(6.4) can be rewritten in the Cartesian coordinates of the rest frame in the following form

$$\begin{aligned}
f(\vec{\eta}(\tau), \vec{\kappa}(\tau), \tau) &= \int \prod_i^{1..N} d^3\eta_i \chi(V) \int \prod_j^{1..N} d^3\kappa_j \\
&\quad \rho(\vec{\eta}_1, \vec{\kappa}_1, \dots, \vec{\eta}_N, \vec{\kappa}_N, \tau) \delta^3(\vec{\mathcal{P}}_N) \delta^3(\vec{\mathcal{P}}_N) \delta^3\left(\frac{\vec{\mathcal{K}}_N}{M_{NC}}\right) \\
&\quad \left[\sum_i \delta^3(\vec{x}(\tau = x_{(cm)}^o) - \vec{x}_{(cm)i}(\tau = x_{(cm)}^o)) \right. \\
&\quad \left. \delta^3(\vec{p}(\tau = x_{(cm)}^o) - \vec{p}_{(cm)i}(\tau = x_{(cm)}^o)) \right] = \\
&= F_{(cm)}(x_{(cm)}^o, \vec{x}_{(cm)}(x_{(cm)}^o), \vec{p}_{(cm)}(x_{(cm)}^o)). \tag{6.6}
\end{aligned}$$

In the free case we can use the results of Subsection C of Section II to rewrite the one-particle distribution function in an arbitrary inertial frame with Cartesian coordinates $x^\mu = \Lambda^\mu{}_\nu x_{(cm)}^\nu$. By using $\tau = x_{(cm)}^o = x_{(cm)i}^o(\tau)$, we get

$$\begin{aligned}
&\sum_i \delta^3(\vec{x}(\tau = x_{(cm)}^o) - \vec{x}_{(cm)i}(\tau = x_{(cm)}^o)) \delta^3(\vec{p}(\tau = x_{(cm)}^o) - \vec{p}_{(cm)i}(\tau = x_{(cm)}^o)) = \\
&= \sum_i \int d\tau_i \frac{dx_{(cm)i}^o(\tau_i)}{d\tau_i} \delta(x_{(cm)}^o - x_{(cm)i}^o(\tau_i)) \delta^3(\vec{x}_{(cm)}(x_{(cm)}^o) - \vec{x}_{(cm)i}(\tau_i)) \\
&\quad \delta^3(\vec{p}_{(cm)}(x_{(cm)}^o) - \vec{p}_{(cm)i}(\tau_i)) = \\
&= \sum_i \int d\tau_i \frac{dx_{(cm)i}^o(\tau_i)}{d\tau_i} \delta^4(x_{(cm)}^\mu - x_{(cm)i}^\mu(\tau_i)) \delta^3(\vec{p}_{(cm)}(x_{(cm)}^o) - \vec{p}_{(cm)i}(\tau_i)). \tag{6.7}
\end{aligned}$$

We used a different τ_i for each particle and we introduced $\frac{dx_{(cm)i}^o(\tau_i)}{d\tau_i} = 1$ (also equal to $\frac{p_{(cm)i}^o}{E_i} = 1$ in the free case).

Then we get the following results:

A) Eq.(2.35) implies

$$\delta^4(x_{(cm)}^\mu - x_{(cm)i}^\mu(\tau_i)) = \delta^4(x_{(cm)}^\mu(x_{(cm)}^o) - (\Lambda^{-1} x_i(\tau_i))^\mu) = \delta^4((\Lambda x_{(cm)}(x_{(cm)}^o))^\mu - x_i^\mu(\tau_i)); \tag{6.8}$$

B) the invariance of $\delta^4(p)$ with $\epsilon p^2 = m^2 c^2$ implies

$$\begin{aligned}
\delta^3(\vec{p}_{(cm)}(x_{(cm)}^o) - \vec{p}_{(cm)i}(\tau_i)) &= \delta^3(\vec{p}_{(cm)}(x_{(cm)}^o) - \Lambda^{-1} p_i(\tau_i)) = \\
&= \frac{p_i^o(\tau_i)}{(\Lambda^{-1} p_i(\tau_i))^o} \delta^3(\Lambda \vec{p}_{(cm)}(x_{(cm)}^o) - \vec{p}_i(\tau_i)); \tag{6.9}
\end{aligned}$$

C) Eq.(2.26) implies

$$\frac{p_i^o(\tau_i)}{(\Lambda^{-1} p_i(\tau_i))^o} = \frac{p_i^o(\tau_i)}{E_i} \frac{dx_{(cm)i}^o(\tau_i)}{d\tau_i} = \frac{dx_i^o(\tau_i)}{d\tau_i}. \quad (6.10)$$

As a consequence, by putting $x^\mu(x^o) = (\Lambda x_{(cm)}(x_{(cm)}^o))^\mu$ and $\vec{p}(x^o) = \vec{\Lambda} p_{(cm)}(x_{(cm)}^o)$ we have

$$\begin{aligned} & \sum_i \int d\tau_i \frac{dx_{(cm)i}^o(\tau_i)}{d\tau_i} \delta^4(x_{(cm)}^\mu - x_{(cm)i}^\mu(\tau_i)) \delta^3(\vec{p}_{(cm)}(x_{(cm)}^o) - \vec{p}_{(cm)i}(\tau_i)) = \\ & = \sum_i \int d\tau_i \frac{dx_i^o(\tau_i)}{d\tau_i} \delta^4(x^\mu(x^o) - x_i^\mu(\tau_i)) \delta^3(\vec{p}(x^o) - \vec{p}_i(\tau_i)). \end{aligned} \quad (6.11)$$

Therefore by using $\chi(V) = \chi(V(\Lambda))$ and after having re-expressed the internal Poincaré generators in terms of $x = \Lambda x_{(cm)}$ and $p = \Lambda p_{(cm)}$, we get that Eq.(6.11) implies

$$\begin{aligned} f(\vec{\eta}(\tau), \vec{\kappa}(\tau), \tau) &= F_{(cm)}(x_{(cm)}^o, \vec{x}_{(cm)}(x_{(cm)}^o), \vec{p}_{(cm)}(x_{(cm)}^o)) = \\ &= F(x^o = ((\Lambda x_{(cm)}(x_{(cm)}^o))^o, \vec{x}(x^o)) = \vec{\Lambda} x_{(cm)}(x_{(cm)}^o), \vec{p}(x^o) = \vec{\Lambda} p_{(cm)}(x_{(cm)}^o)). \end{aligned} \quad (6.12)$$

This means that the one-particle distribution function is a *Lorentz scalar* in our approach. See Refs. [43, 44] for the existing points of view on this topic.

B. On the Relativistic Boltzmann Equation when we Start from a System of N Relativistic Interacting Particles

Let us show which are the problems in trying to apply the method of the BBGKY hierarchy, used in Chapter 3 of Ref.[20] to derive the non-relativistic Boltzmann equation by using the Liouville equation, to an isolated system of N interacting relativistic particles of equal mass m like the one described in Section IV.

Let us consider a simple form of the model of Section IV in which the function $\mathcal{F}(g; \vec{\eta}_l - \vec{\eta}_m)$ appearing in Eq.(4.4) has the form $\mathcal{F}(g; \vec{\eta}_l - \vec{\eta}_m) = \sum_{i,j \neq l} \tilde{\mathcal{F}}(|\vec{\eta}_l - \vec{\eta}_j|)$, so that the potentials appearing in Eq.(4.5) have the form $V_i(g; \vec{\eta}_l - \vec{\eta}_m, \vec{\kappa}_i) = \mathcal{V}_i(|\vec{\eta}_l - \vec{\eta}_m|, \vec{\kappa}_i) = 2 \vec{\kappa}_i \cdot \sum_{k \neq i} \frac{\partial \tilde{\mathcal{F}}(|\vec{\eta}_i - \vec{\eta}_j|)}{\partial \vec{\eta}_i} + \left(\sum_{k \neq i} \frac{\partial \tilde{\mathcal{F}}(|\vec{\eta}_i - \vec{\eta}_j|)}{\partial \vec{\eta}_i} \right)^2$.

In this case like in Ref.[20] we can assume that the density function $\rho(\vec{\eta}_1, \vec{\kappa}_1, \dots, \vec{\eta}_N, \vec{\kappa}_N, \tau)$ is symmetric in the exchange of the particles. Moreover in this case also the functions $\vec{\mathcal{P}}_N$ and $\vec{\mathcal{K}}_N/M_N c$ are symmetric in the exchange of the particles.

If we use the same notation of Ref.[20], namely $z_i = (\vec{\eta}_i, \vec{\kappa}_i)$, $dz_i = d^3\eta_i d^3\kappa_i$, we can rewrite Eq.(6.4) in the form

$$f_1(z_1, \tau) = f(z_1, \tau) = N \int dz_2 \dots dz_N \rho(z_1, z_2, \dots, z_N, \tau) \delta^3(\vec{\mathcal{P}}_N) \delta^3\left(\frac{\vec{\mathcal{K}}_N}{M_N c}\right), \quad (6.13)$$

and we can also define the s-particle distribution functions

$$f_s(z_1, \dots, z_s, \tau) = \frac{N!}{(N-s)!} \int \prod_j^{s+1..N} d^3\eta_j \chi(V) \int dz_{s+1} \dots dz_N \rho(z_1, \dots, z_s, \tau) \delta^3(\vec{\mathcal{P}}_N) \delta^3\left(\frac{\vec{\mathcal{K}}_N}{M_N c}\right). \quad (6.14)$$

The Liouville operator implied by Eq.(4.7) is $(E_i(\tau) = \sqrt{m^2 c^2 + \vec{\kappa}_i^2} + \mathcal{V}_i(|\vec{\eta}_l - \vec{\eta}_m|, \vec{\kappa}_i))$

$$\begin{aligned} \hat{L} &= \frac{\partial}{\partial \tau} + \hat{h}_N, \\ \hat{h}_N &= \sum_i^{1..N} \left[\frac{1}{E_i(\tau)} \left(\kappa_{ir}(\tau) + \sum_{k \neq i} \frac{\partial \tilde{\mathcal{F}}(|\vec{\eta}_i - \vec{\eta}_k|)}{\partial \eta_i^r} \right) \frac{\partial}{\partial \eta_i^r} - \right. \\ &\quad \left. - \sum_j \frac{1}{E_j(\tau)} \frac{\partial \mathcal{V}_j(|\vec{\eta}_l - \vec{\eta}_m|, \vec{\kappa}_i)}{\partial \eta_i^r} \frac{\partial}{\partial \kappa_{ir}} \right], \end{aligned} \quad (6.15)$$

and we get

$$\begin{aligned} \frac{\partial}{\partial \tau} f_s(z_1, \dots, z_s, \tau) &= \\ &= \frac{N!}{(N-s)!} \int dz_{s+1} \dots dz_N \frac{\partial}{\partial \tau} \rho(z_1, \dots, z_N, \tau) \delta^3(\vec{\mathcal{P}}_N) \delta^3\left(\frac{\vec{\mathcal{K}}_N}{M_N c}\right) = \\ &= -\frac{N!}{(N-s)!} \int dz_{s+1} \dots dz_N \left(\hat{h}_N \rho(1, \dots, N, \tau) \right) \delta^3(\vec{\mathcal{P}}_N) \delta^3\left(\frac{\vec{\mathcal{K}}_N}{M_N c}\right). \end{aligned} \quad (6.16)$$

The presence of the terms $E_i(\tau)$, containing potentials depending upon all the $\vec{\eta}_k$ and also on the momentum $\vec{\kappa}_i$, does not allow us to write $\hat{h}_N(z_1, \dots, z_N) = \hat{h}_s(z_1, \dots, z_s) + \hat{h}_{N-s}(z_{s+1}, \dots, z_N) + \sum_{u=1}^s \sum_{v=s+1}^N \hat{P}_{uv}$ (with $\hat{P}_{ij} = \vec{f}(|\vec{\eta}_i - \vec{\eta}_j|) \cdot (\frac{\partial}{\partial \vec{\kappa}_i} - \frac{\partial}{\partial \vec{\kappa}_j})$) like in the non-relativistic case (where $\hat{h}_N = \sum_{i=1}^N \frac{\vec{\kappa}_i}{m} \frac{\partial}{\partial \vec{\eta}_i} + \frac{1}{2} \sum_{i \neq j} \hat{P}_{ij}$) and to have $\int dz_{s+1} \dots dz_N \hat{h}_{N-s}(z_{s+1}, \dots, z_N) \rho(z_1, \dots, z_N, \tau) = 0$.

Therefore we do not get $(\frac{\partial}{\partial \tau} + \hat{h}_s) f_s(z_1, \dots, z_s, \tau) = \dots - \sum_{j=1}^s \int dz_{s+1} \hat{P}_{i,s+1} f_{s+1}(z_1, \dots, z_{s+1}, \tau)$ (Eq.(3.57) of Ref.[20]), namely a set of coupled equations for the distribution functions f_s from which to extract the Boltzmann equation for f_1 by means of the physical approximations described in Ref.[20].

However in the more physically relevant model of N charged particles interacting through the Coulomb plus Darwin potential whose Hamiltonian is given in Eq.(A1), the procedure of Ref.[20] can be used to get a relativistic Boltzmann equation, $(\frac{\partial}{\partial \tau} + \frac{\vec{\kappa}}{\sqrt{m^2 c^2 + \vec{\kappa}^2}} \cdot \frac{\partial}{\partial \vec{\eta}}) f_1(\vec{\eta}, \vec{\kappa}, \tau) = C(\vec{\eta}, \vec{\kappa})$, if the term $V_{Darwin}(\vec{\eta}_i - \vec{\eta}_j, \vec{\kappa}_k)$ corresponding to the Darwin potential (relevant for relativistic bound states) is negligible.

VII. RELATIVISTIC DISSIPATIVE FLUIDS

In equilibrium relativistic statistical mechanics [49] a perfect fluid with internal energy ρ and pressure p (connected by some equation of state) has an energy-momentum tensor $T^{\mu\nu} = \epsilon \rho U^\mu U^\nu - p (\eta^{\mu\nu} - \epsilon U^\mu U^\nu)$, where the unit hydrodynamical 4-velocity U^μ is a time-like eigenvector of the energy-momentum tensor, $\epsilon U_\mu T^{\mu\nu} = \rho U^\nu$. The conserved particle current is $J^\mu = n U^\mu$ with n the particle number density. All the relevant 4-vectors are parallel to U^μ at equilibrium: $P^\mu = p U^\mu$, $V^\mu = V U^\mu$, $\beta^\mu = \beta U^\mu$ ($\beta = 1/k_B T$). In relativistic thermodynamics also the entropy is a 4-vector parallel to U^μ : $S^\mu = S U^\mu = \frac{p}{c^2} \beta^\mu - \alpha J^\mu - \epsilon \beta_\nu T^{\nu\mu}$ ($\alpha = \beta (\mu - k_B T S/n)$ is the thermal potential with μ the chemical potential). At equilibrium one has the Killing equation $\partial_\mu \alpha = \partial_\mu \beta_\nu + \partial_\nu \beta_\mu = 0$.

As shown in Refs.[49, 50] near equilibrium all these 4-vector quantities are no longer parallel to U^μ and $T^{\mu\nu}$ has the decomposition of a viscous fluid. To treat dissipative fluids out of equilibrium one needs a relativistic kinetic theory, in which the one-particle distribution function of the diluted gas is used to obtain a hydrodynamical description of an effective fluid.

In relativistic kinetic theory the one-particle distribution function $f(x, p) = \tilde{f}(\vec{x}, \vec{p}, x^o)$, with the particle energy given by $p^o = \sqrt{m^2 c^2 + \vec{p}^2}$, is used to define:

a) a conserved particle current

$$J^\mu(x) = \int \frac{mc}{h^3} \frac{d^3 p}{p^o} \frac{p^\mu}{mc} f(x, p); \quad (7.1)$$

b) an energy-momentum tensor ²⁶

$$T^{\mu\nu}(x) = \int \frac{mc}{h^3} \frac{d^3 p}{p^o} \frac{p^\mu}{mc} p^\nu f(x, p), \quad \text{with } \epsilon p^2 = m^2 c^2, \quad (7.2)$$

c) an infinite number of higher moments $F^{\mu\nu\alpha\beta\ldots}(x) = \int \frac{mc}{h^3} \frac{d^3 p}{p^o} \frac{p^\mu}{mc} p^\nu p^\alpha p^\beta \ldots f(x, p)$ (in some approximate model one gets a closed description only in terms of 14 of them; see Ref.[62]).

It is assumed that this is the description of a perfect gas with some equation of state like the Boltzmann perfect gas.

Then one defines an entropy 4-vector

$$S^\mu(x) = -k_B \int \frac{mc}{h^3} \frac{d^3 p}{p^o} \frac{p^\mu}{mc} f(x, p) \ln f(x, p). \quad (7.3)$$

and the second law of thermodynamics $\partial_\mu S^\mu(x) \geq 0$ (H-theorem) emerges.

As said in the Introduction this is the standard way to study relativistic dissipative fluids and their transport coefficients. The results of Section VI can be relevant for a sound relativistic treatment of these problems.

²⁶ It has no connection with the $T^{\mu\nu}$ of Eq.(2.13).

On the other hand equilibrium perfect fluids with equation of state $\rho = \rho(n, s)$ (n is the particle number density, s the equilibrium entropy) can be described by means of the action principle of Ref.[15], which has been adapted to the framework of our RCM in Refs.[13, 14, 18]. In this approach the isentropic fluid is described in terms of Lagrangian comoving 3-coordinates $\alpha^i(\tau, \vec{\sigma})$ on Wigner 3-spaces and the action is

$$S = \int d\tau d^3\sigma \rho(n(\alpha^i), s(\alpha^i))(\tau, \vec{\sigma}). \quad (7.4)$$

The particle number density has the form $n(\alpha^i(\tau, \vec{\sigma})) = \sqrt{\epsilon \eta_{AB} J^A(\alpha^i(\tau, \vec{\sigma})) J^B(\alpha^i(\tau, \vec{\sigma}))}$ and the unit 4-velocity is $U^A(\alpha^i(\tau, \vec{\sigma})) = J^A(\alpha^i(\tau, \vec{\sigma}))/n(\alpha^i(\tau, \vec{\sigma}))$. It satisfies the comoving condition $U^A(\alpha^i(\tau, \vec{\sigma})) \partial_A \alpha^i(\tau, \vec{\sigma}) = 0$. This implies the conservation of the particle number, $\partial_A J^A(\alpha^i(\tau, \vec{\sigma})) = \partial_A (n(\alpha^i(\tau, \vec{\sigma})) U^A(\alpha^i(\tau, \vec{\sigma}))) = 0$ and the conservation of entropy (no no entropy exchange between different fluid particles), $\partial_A (s(\alpha^i(\tau, \vec{\sigma})) n(\alpha^i(\tau, \vec{\sigma})) U^A(\alpha^i(\tau, \vec{\sigma}))) = 0$, i.e. $U^A(\alpha^i(\tau, \vec{\sigma})) \partial_A s(\alpha^i(\tau, \vec{\sigma})) = 0$.

As shown in Ref.[15] this description is in accord with standard relativistic thermodynamics: one has the Euler relation $\rho + p = T s + \mu n$ (μ is the chemical potential), the Gibbs relation (first law of thermodynamics) $d\rho = \mu dn + n T ds$ and the second law of thermodynamics at equilibrium, i.e. $U^A(\alpha^i(\tau, \vec{\sigma})) \partial_A s(\alpha^i(\tau, \vec{\sigma})) = 0$.

We want to add a suggestion about the possibility of using our RCM framework to describe relativistic dissipative fluids with an action principle, by relaxing the condition that the fluid is isentropic (see Ref.[63] for what is known about non-isentropic fluids).

If we replace the equilibrium entropy $s(\alpha^i(\tau, \vec{\sigma}))$ with a function $\tilde{s} = \sqrt{\epsilon \eta_{AB} \mathcal{S}^A \mathcal{S}^B}$ built in terms of a 4-vector $\mathcal{S}^A(\alpha^i, \partial_r \alpha^i, \partial_r \partial_s \alpha^i, \dots)(\tau, \vec{\sigma})$, we loose the equilibrium condition, i.e. $U^A(\alpha^i(\tau, \vec{\sigma})) \partial_A \tilde{s}(\alpha^i(\tau, \vec{\sigma}), \partial_r \alpha^i(\tau, \vec{\sigma}), \dots) \neq 0$ due to an entropy exchange among different fluid particles. Following the existing literature on dissipative systems, quoted in the Introduction, the 4-vector \mathcal{S}^A is assumed to have a parametrization of the type

$$\mathcal{S}^A = s U^A + \mathcal{V}^A, \quad U_A \mathcal{V}^A = 0, \quad (7.5)$$

with $\mathcal{V}^A = q^A + \text{viscous terms}$ (q^A is the heat transfer 4-vector).

The new action principle is

$$S = \int d\tau d^3\sigma \left[\rho(n(\alpha^i), \tilde{s}(\alpha^i, \partial_r \alpha^i, \dots))(\tau, \vec{\sigma}) + \lambda(\tau, \vec{\sigma}) \left(\partial_A \mathcal{S}^A - \sum_k \mathcal{A}_k^2(\mathcal{V}^B) \right)(\tau, \vec{\sigma}) \right], \quad (7.6)$$

with $\lambda(\tau, \vec{\sigma})$ a Lagrange multiplier implying the second law of thermodynamics, $\partial_A \mathcal{S}^A(\tau, \vec{\sigma}) \geq 0$.

The functions $\mathcal{A}_k^2(\mathcal{V}^B)(\tau, \vec{\sigma})$ can be taken to have the form resulting from the structure of $\partial_A \mathcal{S}^A(\tau, \vec{\sigma})$ in the models existing in the literature (like the Eckart or the Israel-Stewart models). In this way we could have both a Lagrangian and Hamiltonian formulation of the equations of motion of dissipative systems and study the difference between the cases in which the Lagrange multiplier remains arbitrary (first class constraints at the Hamiltonian

level) and those in which it turns out to be determined by the dynamics (second class constraints at the Hamiltonian level). In some cases it could also happen that the assumed functions $\mathcal{A}_k^2(\mathcal{V}^B)(\tau, \vec{\sigma})$ are incompatible with the dynamics leading to contradictions. These problems will be studied in a future paper.

Let us add a final remark. The previous formulation of the action principle is based on the admissible 3+1 splittings of Minkowski space-time of Section II. In non-inertial rest frames we have the unit normal to the 3-spaces $l^\mu(\tau, \vec{\sigma})$ to be used as a unit 4-velocity (the congruence of the time-like Eulerian observers) as an alternative to both "Eckart theory" (4-velocity parallel to the particle current ²⁷) and "Landau-Lifshitz theory" (4-velocity as the time-like eigenvector of the energy-momentum tensor).

²⁷ In the 3+1 formalism it is connected to the skew congruence of time-like observers with 4-velocity $z^\mu_\tau(\tau, \vec{\sigma})$. See Ref.[18] for the properties of the two congruences of observers in the case of dust: in general the dust 4-velocity is different from both these 4-velocities.

VIII. CONCLUSIONS

Our new consistent RCM allowed us to revisit long standing problems in relativistic statistical mechanics and relativistic kinetic theory.

The first part of the paper is devoted to a clarification of the kinematical background which was the basic tool to develop a relativistic theory of isolated systems in accord with what is known about relativistic bound states and using a methodology for the definition of the instantaneous 3-spaces which is a mathematical idealization of the protocols for clock synchronization used in atomic and space physics. In this way we can arrive at a description of N-particle systems with various kinds of interactions and with an explicit knowledge of the Poincaré generators and a control on the relativistic collective variables in Minkowski space-time. The non-relativistic limit allows to recover a Hamilton-Jacobi description of the Galilei generators in Galilei space-time.

After a study of the inertial rest frames of isolated systems we also gave their description in the non-inertial rest frames, both at the relativistic and non-relativistic level.

The main result of the paper is the use of this framework to give a definition of the micro-canonical ensemble both in relativistic and non-relativistic statistical mechanics which utilizes only the Poincaré or Galilei generators. The definition is given in the inertial rest frames, but it can be extended to the non-inertial ones (till now no such an extension was known). The so defined non-inertial micro-canonical ensemble turns out to describe an equilibrium configuration notwithstanding the presence of long-range inertial potentials due to the fact the asymptotic Poincaré or Galilei generators are constants of the motion in non-inertial rest frames.

With our definition of the relativistic micro-canonical ensemble the micro-canonical temperature turns out to be a Lorentz scalar. Since we do not discriminate between long- and short-range interactions, we have no statement about the relativistic canonical ensemble, except that when the thermodynamic limit exists also the canonical temperature is a Lorentz scalar. We have no new statement about the definition of relativistic thermodynamics.

Then we have given a definition of the one-particle distribution function in non-equilibrium relativistic kinetic theory by using a density function for a relativistic Gibbs ensemble with the same transformation properties of the relativistic micro-canonical distribution function. It turns out that this one-particle distribution function is a Lorentz scalar in our approach. Then we show that the form of the Poincaré generators for N interacting particles in general prevents to copy the non-relativistic derivation of the Boltzmann equation based on the BBGKY hierarchy except in special cases and with drastic approximations.

Finally we made some comments on relativistic dissipative fluids in the framework of inertial and non-inertial rest frames, an argument relevant for relativistic heavy ion collisions and for cosmology. In particular we suggest the possibility to give their description by means of an action principle with the second law of thermodynamics enforced with a Lagrange multiplier. Future work will be needed to see whether it is a constructive proposal.

In conclusion we begin to understand the complications of relativistic physics, complications induced by the Lorentz signature of Minkowski space-time and which disappear in the non-relativistic limit.

Appendix A: N Charged Scalar Particles with Coulomb plus Darwin Potentials

The internal Poincaré generators of a system of N charged scalar particles (with Grassmann-valued electric charges to regularize the electro-magnetic self-energies) interacting with the Coulomb plus Darwin potentials were evaluated in Ref.[4]. They have the following expression

$$\begin{aligned}\mathcal{E}_{(int)} &= M c^2 = c \sum_{i=1}^N \sqrt{m_i^2 c^2 + \vec{\kappa}_i^2} + \sum_{i \neq j}^{1..N} \frac{Q_i Q_j}{4\pi |\vec{\eta}_i - \vec{\eta}_j|} + V_{DARWIN}, \\ V_{DARWIN} &= \sum_{i \neq j}^{1..N} Q_i Q_j \left(\frac{\vec{\kappa}_i \cdot \vec{A}_{\perp Sj}(\tau, \vec{\eta}_i(\tau))}{\sqrt{m_i^2 c^2 + \vec{\kappa}_i^2}} + \right. \\ &\quad + \int d^3\sigma \left[\frac{1}{2} (\vec{\pi}_{\perp Si} \cdot \vec{\pi}_{\perp Sj} + \vec{B}_{Si} \cdot \vec{B}_{Sj}) + \right. \\ &\quad \left. \left. + \left(\frac{\vec{\kappa}_i}{\sqrt{m_i^2 c^2 + \vec{\kappa}_i^2}} \cdot \frac{\partial}{\partial \vec{\eta}_i} \right) (\vec{A}_{\perp Si} \cdot \vec{\pi}_{\perp Sj} - \vec{\pi}_{\perp Si} \cdot \vec{A}_{\perp Sj}) \right] (\tau, \vec{\sigma}) \right).\end{aligned}$$

$$\begin{aligned}\vec{\mathcal{P}}_{(int)} &= \sum_{i=1}^N \tilde{\vec{\kappa}}_i \approx 0, \\ \vec{\mathcal{J}}_{(int)} &= \sum_{i=1}^N \tilde{\vec{\eta}}_i \times \tilde{\vec{\kappa}}_i,\end{aligned}$$

$$\begin{aligned}\vec{\mathcal{K}}_{(int)} &= - \sum_{i=1}^N \vec{\eta}_i(\tau) \left[\sqrt{m_i^2 c^2 + \vec{\kappa}_i^2} + \right. \\ &\quad + \frac{\vec{\kappa}_i}{2 c \sqrt{m_i^2 c^2 + \vec{\kappa}_i^2}} \cdot \sum_{j \neq i}^{1..N} Q_i Q_j \left(\frac{1}{2} \frac{\partial \mathcal{K}_{ij}(\vec{\kappa}_i, \vec{\kappa}_j, \vec{\eta}_i - \vec{\eta}_j)}{\partial \vec{\eta}_i} - 2 \vec{A}_{\perp Sj}(\vec{\kappa}_j, \vec{\eta}_i - \vec{\eta}_j) \right) S \Big] - \\ &\quad - \frac{1}{2} \sum_{i=1}^N \sum_{j \neq i}^{1..N} \frac{Q_i Q_j}{c} \sqrt{m_i^2 c^2 + \vec{\kappa}_i^2} \frac{\partial \mathcal{K}_{ij}(\vec{\kappa}_i, \vec{\kappa}_j, \vec{\eta}_i - \vec{\eta}_j)}{\partial \vec{\kappa}_i} + \\ &\quad + \sum_{i=1}^N \sum_{j \neq i}^{1..N} \frac{Q_i Q_j}{8\pi c} \frac{\vec{\eta}_i - \vec{\eta}_j}{|\vec{\eta}_i - \vec{\eta}_j|} - \sum_{i=1}^N \sum_{j \neq i}^{1..N} \frac{Q_i Q_j}{4\pi c} \int d^3\sigma \frac{\vec{\pi}_{\perp Sj}(\vec{\sigma} - \vec{\eta}_j, \vec{\kappa}_j)}{|\vec{\sigma} - \vec{\eta}_i|} - \\ &\quad - \frac{1}{2c} \sum_{i=1}^N \sum_{j \neq i}^{1..N} Q_i Q_j \int d^3\sigma \vec{\sigma} \left[\vec{\pi}_{\perp Si}(\vec{\sigma} - \vec{\eta}_i, \vec{\kappa}_i) \cdot \vec{\pi}_{\perp Sj}(\vec{\sigma} - \vec{\eta}_j, \vec{\kappa}_j) + \right. \\ &\quad \left. + \vec{B}_{Si}(\vec{\sigma} - \vec{\eta}_i, \vec{\kappa}_i) \cdot \vec{B}_{Sj}(\vec{\sigma} - \vec{\eta}_j, \vec{\kappa}_j) \right] \approx 0,\end{aligned}\tag{A1}$$

with $Q_i Q_j \mathcal{K}_{ij}(\tau) = Q_i Q_j \int d^3\sigma \left[\vec{A}_{\perp Si} \cdot \vec{\pi}_{\perp Sj} - \vec{\pi}_{\perp Si} \cdot \vec{A}_{\perp Sj} \right] (\tau, \vec{\sigma})$.

These generators depend upon the Lienard-Wiechert transverse electromagnetic potential, electric field and magnetic field, inhomogeneous solution of the equations $\square A_\perp^r(\tau, \vec{\sigma}) \doteq j_\perp^r(\tau, \vec{\sigma}) = \sum_i Q_i P_\perp^{rs}(\vec{\sigma}) \dot{\eta}_i^s(\tau) \delta^3(\vec{\sigma} - \vec{\eta}_i(\tau))$, which were evaluated in Ref.[8] (and extended to spinning particles in Ref.[9]). The functions $\vec{A}_{\perp S}(\tau, \vec{\sigma})$, $\vec{E}_{\perp S}(\tau, \vec{\sigma}) = \vec{\pi}_{\perp S}(\tau, \vec{\sigma})$ and $\vec{B}_S(\tau, \vec{\sigma})$ have the following expression

$$\begin{aligned} \vec{A}_{\perp S}(\tau, \vec{\sigma}) &\doteq \sum_{i=1}^N Q_i \vec{A}_{\perp Si}(\vec{\sigma} - \vec{\eta}_i(\tau), \vec{\kappa}_i(\tau)), \\ \vec{A}_{\perp Si}(\vec{\sigma} - \vec{\eta}_i, \vec{\kappa}_i) &= \frac{1}{4\pi|\vec{\sigma} - \vec{\eta}_i|} \frac{1}{\sqrt{m_i^2 c^2 + \vec{\kappa}_i^2} + \sqrt{m_i^2 c^2 + (\vec{\kappa}_i \cdot \frac{\vec{\sigma} - \vec{\eta}_i}{|\vec{\sigma} - \vec{\eta}_i|})^2}} \times \\ &\left[\vec{\kappa}_i + \frac{[\vec{\kappa}_i \cdot (\vec{\sigma} - \vec{\eta}_i)] (\vec{\sigma} - \vec{\eta}_i)}{|\vec{\sigma} - \vec{\eta}_i|^2} \frac{\sqrt{m_i^2 c^2 + \vec{\kappa}_i^2}}{\sqrt{m_i^2 c^2 + (\vec{\kappa}_i \cdot \frac{\vec{\sigma} - \vec{\eta}_i}{|\vec{\sigma} - \vec{\eta}_i|})^2}} \right], \end{aligned} \quad (\text{A2})$$

$$\begin{aligned} \vec{E}_{\perp S}(\tau, \vec{\sigma}) &= \vec{\pi}_{\perp S}(\tau, \vec{\sigma}) = -\frac{\partial \vec{A}_{\perp S}(\tau, \vec{\sigma})}{\partial \tau} = \\ &= \sum_{i=1}^N Q_i \vec{\pi}_{\perp Si}(\vec{\sigma} - \vec{\eta}_i(\tau), \vec{\kappa}_i(\tau)) = \\ &= \sum_{i=1}^N Q_i \frac{\vec{\kappa}_i(\tau) \cdot \vec{\partial}_\sigma}{\sqrt{m_i^2 c^2 + \vec{\kappa}_i^2(\tau)}} \vec{A}_{\perp Si}(\vec{\sigma} - \vec{\eta}_i(\tau), \vec{\kappa}_i(\tau)) = \\ &= -\sum_{i=1}^N Q_i \times \\ &\frac{1}{4\pi|\vec{\sigma} - \vec{\eta}_i(\tau)|^2} \left[\vec{\kappa}_i(\tau) [\vec{\kappa}_i(\tau) \cdot \frac{\vec{\sigma} - \vec{\eta}_i(\tau)}{|\vec{\sigma} - \vec{\eta}_i(\tau)|}] \frac{\sqrt{m_i^2 c^2 + \vec{\kappa}_i^2(\tau)}}{[m_i^2 c^2 + (\vec{\kappa}_i(\tau) \cdot \frac{\vec{\sigma} - \vec{\eta}_i(\tau)}{|\vec{\sigma} - \vec{\eta}_i(\tau)|})^2]^{3/2}} + \right. \\ &+ \frac{\vec{\sigma} - \vec{\eta}_i(\tau)}{|\vec{\sigma} - \vec{\eta}_i(\tau)|} \left(\frac{\vec{\kappa}_i^2(\tau) + (\vec{\kappa}_i(\tau) \cdot \frac{\vec{\sigma} - \vec{\eta}_i(\tau)}{|\vec{\sigma} - \vec{\eta}_i(\tau)|})^2}{\vec{\kappa}_i^2(\tau) - (\vec{\kappa}_i(\tau) \cdot \frac{\vec{\sigma} - \vec{\eta}_i(\tau)}{|\vec{\sigma} - \vec{\eta}_i(\tau)|})^2} \left(\frac{\sqrt{m_i^2 c^2 + \vec{\kappa}_i^2(\tau)}}{\sqrt{m_i^2 c^2 + (\vec{\kappa}_i(\tau) \cdot \frac{\vec{\sigma} - \vec{\eta}_i(\tau)}{|\vec{\sigma} - \vec{\eta}_i(\tau)|})^2}} - 1 \right) + \right. \\ &\left. \left. + \frac{(\vec{\kappa}_i(\tau) \cdot \frac{\vec{\sigma} - \vec{\eta}_i(\tau)}{|\vec{\sigma} - \vec{\eta}_i(\tau)|})^2 \sqrt{m_i^2 c^2 + \vec{\kappa}_i^2(\tau)}}{[m_i^2 c^2 + (\vec{\kappa}_i(\tau) \cdot \frac{\vec{\sigma} - \vec{\eta}_i(\tau)}{|\vec{\sigma} - \vec{\eta}_i(\tau)|})^2]^{3/2}} \right) \right], \end{aligned} \quad (\text{A3})$$

$$\begin{aligned} \vec{B}_S(\tau, \vec{\sigma}) &= \vec{\partial} \times \vec{A}_{\perp S}(\tau, \vec{\sigma}) = \sum_{i=1}^N Q_i \vec{B}_{Si}(\vec{\sigma} - \vec{\eta}_i(\tau), \vec{\kappa}_i(\tau)) = \\ &= \sum_{i=1}^N Q_i \frac{1}{4\pi|\vec{\sigma} - \vec{\eta}_i(\tau)|^2} \frac{m_i^2 c^2 \vec{\kappa}_i(\tau) \times \frac{\vec{\sigma} - \vec{\eta}_i(\tau)}{|\vec{\sigma} - \vec{\eta}_i(\tau)|}}{[m_i^2 c^2 + (\vec{\kappa}_i(\tau) \cdot \frac{\vec{\sigma} - \vec{\eta}_i(\tau)}{|\vec{\sigma} - \vec{\eta}_i(\tau)|})^2]^{3/2}}. \end{aligned} \quad (\text{A4})$$

Appendix B: N Scalar Particles in Newtonian Gravity and in Post-Minkowskian Tetrad Gravity

Let us now consider a system of N particles with mutual gravitational interaction both in Newtonian gravity and in the Hamiltonian Post-Minkowskian (HPM) linearization of ADM tetrad gravity defined in the three papers of Ref.[17].

1. The Galilei Generators for Newtonian Gravity

In Ref.[29] there is the definition of the ordinary and extended micro-canonical ensembles in inertial frames of Galilei space-time in the case of the long-range interactions of Newtonian gravity. The Galilei generators (not in the rest frame) for N particles with the same mass m have the following expression

$$\begin{aligned} E &= H = \sum_{i=1}^N \frac{\vec{p}_i^2}{2m} - G m^2 \sum_{i<j} \frac{1}{\vec{x}_i - \vec{x}_j}, \\ \vec{P} &= \sum_{i=1}^N \vec{p}_i, \\ \vec{J} &= \sum_{i=1}^N \vec{x}_i \times \vec{p}_i, \\ \vec{K} &= t \sum_{i=1}^N \vec{p}_i - \frac{1}{N} \sum_{i=1}^N \vec{x}_i. \end{aligned} \tag{B1}$$

2. The Internal Weak ADM Poincaré' Generators in Hamiltonian Post-Minkowskian Gravity

Let us now consider N particles in Einstein gravity by using its ADM formulation extended to ADM tetrad gravity (for a future inclusion of spinning particles and fermionic fields). In Ref.[17] this theory is fully developed in asymptotically Minkowskian space-times without super-translations, in which it is possible to use 3+1 splittings like the ones of Section II to define the instantaneous 3-spaces. In this class of space-times it is possible to define asymptotic ADM Poincaré generators, using point particles and the electro-magnetic field as matter. By turning off the Newton constant these generators collapse in the ones of Appendix A in Minkowski space-time. The absence of super-translations implies that the instantaneous 3-spaces are non-inertial rest frames of the 3-universe. The asymptotic Minkowski metric at spatial infinity is used as an asymptotic background.

Now also the sign of the energy of the particles is Grassmann-valued to regularize the gravitational self-energies. In the second paper of Ref.[17] there is a HPM linearization of the theory with the identification of PM gravitational waves (GW) described by functions $\Gamma_a^{(1)}(\tau, \vec{\sigma})$ with $\sum_{a=1}^3 \Gamma_a^{(1)}(\tau, \vec{\sigma}) = 0$. One has to put a ultraviolet cutoff on the matter for consistency.

The HPM linearization is done in 3-orthogonal Schwinger time gauges in which the 3-metric of the 3-spaces is diagonal. This fixes the 3-coordinates on the 3-spaces. The only left gauge freedom is an inertial potential ${}^3K_{(1)}(\tau, \vec{\sigma})$, the York time, namely the trace of the extrinsic curvature of the 3-spaces as 3-sub-manifolds of the space-time.

The HPM linearization of the ADM Poincare' generators is given in Eqs. (4-21)-(4.24) of paper II of Ref.[17]. The final expression of the generators in absence of the electro-magnetic field is ²⁸

$$\begin{aligned}
\frac{1}{c} \hat{E}_{ADM} &\approx M_{(1)} c + \frac{1}{c} \hat{E}_{ADM(2)} = \\
&= \sum_i \eta_i \sqrt{m_i^2 c^2 + \vec{\kappa}_i^2(\tau)} - \\
&- \frac{G}{c^3} \sum_{i>j} \eta_i \eta_j \frac{\sqrt{m_i^2 c^2 + \vec{\kappa}_i^2(\tau)} \sqrt{m_j^2 c^2 + \vec{\kappa}_j^2(\tau)}}{|\vec{\eta}_i(\tau) - \vec{\eta}_j(\tau)|} + \frac{G}{c^3} \sum_{i>j} \eta_i \eta_j \left(\frac{4 \vec{\kappa}_i(\tau) \cdot \vec{\kappa}_j(\tau)}{|\vec{\eta}_i(\tau) - \vec{\eta}_j(\tau)|} - \right. \\
&- \left. \frac{1}{4\pi} \sum_{rs} \kappa_{ir}(\tau) \kappa_{js}(\tau) \int d^3\sigma \frac{(\sigma^r - \eta_i^r(\tau)) (\sigma^s - \eta_j^s(\tau))}{|\vec{\sigma} - \vec{\eta}_i(\tau)|^3 |\vec{\sigma} - \vec{\eta}_j(\tau)|^3} \right) - \\
&- \sum_i \eta_i \frac{\vec{\kappa}_i^2(\tau) \left[\frac{G}{c^3} \sum_{j \neq i} \eta_j \frac{\sqrt{m_j^2 c^2 + \vec{\kappa}_j^2(\tau)}}{|\vec{\eta}_i(\tau) - \vec{\eta}_j(\tau)|} + \sum_c \frac{\partial_c^2}{2\Delta} \Gamma_c^{(1)}(\tau, \vec{\eta}_i(\tau)) \right] + \sum_c \kappa_{ic}^2(\tau) \Gamma_c^{(1)}(\tau, \vec{\eta}_i(\tau))}{\sqrt{m_i^2 c^2 + \vec{\kappa}_i^2(\tau)}} + \\
&+ \frac{c^3}{16\pi G} \sum_{\bar{a}\bar{b}} \int d^3\sigma \left[\partial_\tau R_{\bar{a}} M_{\bar{a}\bar{b}} \partial_\tau R_{\bar{b}} + \sum_a \partial_a R_{\bar{a}} M_{\bar{a}\bar{b}} \partial_a R_{\bar{b}} \right] (\tau, \vec{\sigma}) - \\
&- \sum_i \eta_i \vec{\kappa}_i(\tau) \cdot \vec{\partial}^3 \mathcal{K}_{(1)}(\tau, \vec{\eta}_i(\tau)), \tag{B2}
\end{aligned}$$

$$\begin{aligned}
\hat{P}_{ADM}^r &\approx p_{(1)}^r + p_{(2)}^r = \\
&= \sum_i \eta_i \kappa_{ir}(\tau) - \frac{c^3}{8\pi G} \int d^3\sigma \sum_{\bar{a}\bar{b}} \left(\partial_r R_{\bar{a}} M_{\bar{a}\bar{b}} \partial_\tau R_{\bar{b}} \right) (\tau, \vec{\sigma}) + \\
&+ \sum_i \eta_i \sum_a \kappa_{ia}(\tau) \frac{\partial_r \partial_a}{\Delta} \left(\sum_c \frac{\partial_c^2}{2\Delta} \Gamma_c^{(1)} - 2 \Gamma_a^{(1)} \right) (\tau, \vec{\eta}_i(\tau)) - \\
&- \sum_i \eta_i \sqrt{m_i^2 c^2 + \vec{\kappa}_i^2(\tau)} \partial_r {}^3\mathcal{K}_{(1)}(\tau, \vec{\eta}_i(\tau)) \approx 0, \tag{B3}
\end{aligned}$$

²⁸ $\sqrt{m_i^2 c^2 + \vec{\kappa}_i^2(\tau)} = m_i c / \sqrt{1 - \dot{\eta}_i^2(\tau)}$; the spatial operator $M_{\bar{a}\bar{b}}$ is given in Eq.(2.6) of the second paper of Ref.[17]; L is the GW wavelength of Section III of the second paper of Ref.[17]; ${}^3\mathcal{K}(\tau, \vec{\sigma}) = \frac{1}{\Delta} {}^3K(\tau, \vec{\sigma})$.

$$\begin{aligned}
\hat{J}_{ADM}^{rs} &= j_{(1)}^{rs} + j_{(2)}^{rs} = \\
&= \sum_i \eta_i \left(\eta_i^r(\tau) \kappa_{is}(\tau) - \eta_i^s(\tau) \kappa_{ir}(\tau) \right) - \\
&- 2 \sum_i \eta_i \sum_u \kappa_{iu}(\tau) \left(\eta_i^r(\tau) \frac{\partial}{\partial \eta_i^s} - \eta_i^s(\tau) \frac{\partial}{\partial \eta_i^r} \right) \\
&\quad \frac{\partial_u}{\Delta} \left(\Gamma_u^{(1)}(\tau, \vec{\eta}_i(\tau)) - \frac{1}{4} \sum_c \frac{\partial_c^2}{\Delta} \Gamma_c^{(1)} \right) + \\
&+ 2 \sum_i \eta_i \left[\kappa_{ir}(\tau) \frac{\partial_s}{\Delta} \left(\Gamma_s^{(1)} - \frac{1}{4} \sum_c \frac{\partial_c^2}{\Delta} \Gamma_c^{(1)} \right) - \right. \\
&- \left. \kappa_{is}(\tau) \frac{\partial_r}{\Delta} \left(\Gamma_r^{(1)} - \frac{1}{4} \sum_c \frac{\partial_c^2}{\Delta} \Gamma_c^{(1)} \right) \right] (\tau, \vec{\eta}_i(\tau)) - \\
&- \frac{c^3}{8\pi G} \int d^3\sigma \left[\sum_{\bar{a}\bar{b}} (\sigma^r \partial_s - \sigma^s \partial_r) R_{\bar{a}} M_{\bar{a}\bar{b}} \partial_\tau R_{\bar{b}} + 2 {}^3K_{(1)} \partial_r \partial_s (\Gamma_s^{(1)} - \Gamma_r^{(1)}) + \right. \\
&+ \left. 2 (\partial_\tau \Gamma_r^{(1)} + \partial_\tau \Gamma_s^{(1)} - \frac{1}{2} \sum_c \frac{\partial_c^2}{\Delta} \partial_\tau \Gamma_c^{(1)}) \frac{\partial_r \partial_s}{\Delta} (\Gamma_s^{(1)} - \Gamma_r^{(1)}) \right] (\tau, \vec{\sigma}) + \\
&+ \sum_i \eta_i \sqrt{m_i^2 c^2 + \vec{\kappa}_i^2(\tau)} \left(\eta_i^r(\tau) \frac{\partial}{\partial \eta_i^s} - \eta_i^s(\tau) \frac{\partial}{\partial \eta_i^r} \right) {}^3\mathcal{K}_{(1)}(\tau, \vec{\eta}_i(\tau)),
\end{aligned} \tag{B4}$$

$$\begin{aligned}
\hat{J}_{ADM}^{\tau r} &= -\hat{J}_{ADM}^{r\tau} \approx j_{(1)}^{\tau r} + j_{(2)}^{\tau r} = \\
&= - \sum_i \eta_i \eta_i^r(\tau) \sqrt{m_i^2 c^2 + \vec{\kappa}_i^2(\tau)} - \\
&- \frac{G}{c^3} \sum_{i \neq j} \eta_i \eta_j \frac{\vec{\kappa}_i^2(\tau) \sqrt{m_j^2 c^2 + \vec{\kappa}_j^2(\tau)}}{\sqrt{m_i^2 c^2 + \vec{\kappa}_i^2(\tau)}} \frac{\eta_i^r(\tau)}{|\vec{\eta}_i(\tau) - \vec{\eta}_j(\tau)|} + \\
&+ \sum_i \eta_i \eta_i^r(\tau) \sum_a \frac{\kappa_{ia}^2(\tau) \left(\Gamma_a^{(1)} + \frac{1}{2} \sum_c \frac{\partial_c^2}{\Delta} \Gamma_c^{(1)} \right) (\tau, \vec{\eta}_i(\tau))}{\sqrt{m_i^2 c^2 + \vec{\kappa}_i^2(\tau)}} - \\
&- \int d^3\sigma \sigma^r \left[\frac{c^3}{16\pi G} \sum_{a,b} (\partial_a \Gamma_b^{(1)}(\tau, \vec{\sigma}))^2 - \right. \\
&- \frac{c^3}{8\pi G} \sum_a \partial_a \Gamma_a^{(1)}(\tau, \vec{\sigma}) \partial_a \left(\Gamma_a^{(1)} - \frac{1}{2} \sum_c \frac{\partial_c^2}{\Delta} \Gamma_c^{(1)} \right) (\tau, \vec{\sigma}) - \\
&- \left. \frac{c^3}{32\pi G} \sum_a \partial_a \left(\sum_c \frac{\partial_c^2}{\Delta} \Gamma_c^{(1)}(\tau, \vec{\sigma}) \right) \partial_a \left(\sum_d \frac{\partial_d^2}{\Delta} \Gamma_d^{(1)}(\tau, \vec{\sigma}) \right) + \right.
\end{aligned}$$

$$\begin{aligned}
& + \frac{1}{2} \sum_a \sum_i \eta_i \sqrt{m_i^2 c^2 + \vec{\kappa}_i^2(\tau)} \frac{\sigma^a - \eta_i^a(\tau)}{4\pi |\vec{\sigma} - \vec{\eta}_i(\tau)|^3} \partial_a \left(\Gamma_a^{(1)} - \frac{1}{2} \sum_c \frac{\partial_c^2}{\Delta} \Gamma_c^{(1)}(\tau, \vec{\sigma}) \right) + \\
& + \frac{2G}{\pi c^3} \sum_{i \neq j} \eta_i \eta_j \sqrt{m_i^2 c^2 + \vec{\kappa}_i^2(\tau)} \sqrt{m_j^2 c^2 + \vec{\kappa}_j^2(\tau)} \frac{(\vec{\sigma} - \vec{\eta}_i(\tau)) \cdot (\vec{\sigma} - \vec{\eta}_j(\tau))}{|\vec{\sigma} - \vec{\eta}_i(\tau)|^3 |\vec{\sigma} - \vec{\eta}_j(\tau)|^3} + \\
& + \frac{c^3}{16\pi G} \sum_{a,b} \left(\widetilde{M}_{ab}(\vec{\sigma}) \partial_\tau \Gamma_b^{(1)}(\tau, \vec{\sigma}) \right)^2 + \\
& + \frac{c^3}{16\pi G} \sum_{a \neq b} \left[\frac{\partial_a \partial_b \partial_\tau}{\Delta} \left(\Gamma_a^{(1)} + \Gamma_b^{(1)} - \frac{1}{2} \sum_c \frac{\partial_c^2}{\Delta} \Gamma_c^{(1)} \right) (\tau, \vec{\sigma}) \right]^2 - \\
& - \frac{1}{2\pi} \sum_{a,b} \left(\widetilde{M}_{ab}(\vec{\sigma}) \partial_\tau \Gamma_b^{(1)}(\tau, \vec{\sigma}) \right) \sum_i \eta_i \frac{\kappa_{ia}(\tau) (\sigma^a - \eta_i^a(\tau))}{|\vec{\sigma} - \vec{\eta}_i(\tau)|^3} + \\
& + \frac{1}{2\pi} \sum_{a \neq b} \frac{\partial_a \partial_b \partial_\tau}{\Delta} \left(\Gamma_a^{(1)} + \Gamma_b^{(1)} - \frac{1}{2} \sum_c \frac{\partial_c^2}{\Delta} \Gamma_c^{(1)} \right) (\tau, \vec{\sigma}) \sum_i \eta_i \frac{\kappa_{ia}(\tau) (\sigma^a - \eta_i^a(\tau))}{|\vec{\sigma} - \vec{\eta}_i(\tau)|^3} - \\
& - \frac{c^3}{8\pi G} \sum_{a,b} \left(\widetilde{M}_{ab}(\vec{\sigma}) \partial_\tau \Gamma_b^{(1)}(\tau, \vec{\sigma}) \right) \frac{\partial_a^2}{\Delta} \left({}^3K_{(1)}(\tau, \vec{\sigma}) - \frac{G}{c^3} \sum_i \eta_i \frac{\vec{\kappa}_i(\tau) \cdot (\vec{\sigma} - \vec{\eta}_i(\tau))}{|\vec{\sigma} - \vec{\eta}_i(\tau)|^3} \right) + \\
& + \frac{c^3}{8\pi G} \sum_{a \neq b} \frac{\partial_a \partial_b \partial_\tau}{\Delta} \left(\Gamma_a^{(1)} + \Gamma_b^{(1)} - \frac{1}{2} \sum_c \frac{\partial_c^2}{\Delta} \Gamma_c^{(1)} \right) (\tau, \vec{\sigma}) \\
& \quad \frac{\partial_a \partial_b}{\Delta} \left({}^3K_{(1)}(\tau, \vec{\sigma}) - \frac{G}{c^3} \sum_i \eta_i \frac{\vec{\kappa}_i(\tau) \cdot (\vec{\sigma} - \vec{\eta}_i(\tau))}{|\vec{\sigma} - \vec{\eta}_i(\tau)|^3} \right) + \\
& + \frac{c^3}{16\pi G} \sum_{a,b} \left[\frac{\partial_a \partial_b}{\Delta} \left({}^3K_{(1)}(\tau, \vec{\sigma}) - \frac{G}{c^3} \sum_i \eta_i \frac{\vec{\kappa}_i(\tau) \cdot (\vec{\sigma} - \vec{\eta}_i(\tau))}{|\vec{\sigma} - \vec{\eta}_i(\tau)|^3} \right) \right]^2 + \\
& + \frac{1}{2\pi} \sum_{a,b} \sum_i \eta_i \frac{\kappa_{ib}(\tau) (\sigma^a - \eta_i^a(\tau))}{|\vec{\sigma} - \vec{\eta}_i(\tau)|^3} \frac{\partial_a \partial_b}{\Delta} \left({}^3K_{(1)}(\tau, \vec{\sigma}) - \frac{G}{c^3} \sum_j \eta_j \frac{\vec{\kappa}_j(\tau) \cdot (\vec{\sigma} - \vec{\eta}_j(\tau))}{|\vec{\sigma} - \vec{\eta}_j(\tau)|^3} \right) - \\
& - \frac{c^3}{48\pi G} \left({}^3K_{(1)}(\tau, \vec{\sigma}) + \frac{3G}{c^3} \sum_i \eta_i \frac{\vec{\kappa}_i(\tau) \cdot (\vec{\sigma} - \vec{\eta}_i(\tau))}{|\vec{\sigma} - \vec{\eta}_i(\tau)|^3} \right)^2 - \frac{c^3}{24\pi G} \left({}^3K_{(1)}(\tau, \vec{\sigma}) \right)^2 + \\
& + \frac{G}{2\pi c^3} \sum_{i \neq j} \eta_i \eta_j \frac{\vec{\kappa}_i(\tau) \cdot \vec{\kappa}_j(\tau) (\vec{\sigma} - \vec{\eta}_j(\tau)) \cdot (\vec{\sigma} - \vec{\eta}_i(\tau)) + \vec{\kappa}_i(\tau) \cdot (\vec{\sigma} - \vec{\eta}_j(\tau)) \vec{\kappa}_j(\tau) \cdot (\vec{\sigma} - \vec{\eta}_i(\tau))}{|\vec{\sigma} - \vec{\eta}_i(\tau)|^3 |\vec{\sigma} - \vec{\eta}_j(\tau)|^3} \Big] +
\end{aligned}$$

$$\begin{aligned}
& + \int d^3\sigma \left[\frac{3}{8\pi} \sum_i \eta_i \frac{\sqrt{m_i^2 c^2 + \vec{\kappa}_i^2(\tau)}}{|\vec{\sigma} - \vec{\eta}_i(\tau)|} \partial_r \Gamma_r^{(1)} + \frac{3c^3}{16\pi G} \partial_r \Gamma_r^{(1)} \left(\sum_c \frac{\partial_c^2}{\Delta} \Gamma_c^{(1)} \right) \right] (\tau, \vec{\sigma}) + \\
& - \int d^3\sigma \partial_r \left[\frac{c^3}{16\pi G} \left[2 \left(\Gamma_r^{(1)}(\tau, \vec{\sigma}) \right)^2 - \sum_s \left(\Gamma_s^{(1)}(\tau, \vec{\sigma}) \right)^2 - \frac{1}{2} \left(\sum_c \frac{\partial_c^2}{\Delta} \Gamma_c^{(1)}(\tau, \vec{\sigma}) \right)^2 \right] - \right. \\
& \left. - \frac{G}{8\pi c^3} \sum_{i \neq j} \eta_i \eta_j \frac{\sqrt{m_i^2 c^2 + \vec{\kappa}_i^2(\tau)} \sqrt{m_j^2 c^2 + \vec{\kappa}_j^2(\tau)}}{|\vec{\sigma} - \vec{\eta}_i(\tau)| |\vec{\sigma} - \vec{\eta}_j(\tau)|} \right] \approx 0.
\end{aligned} \tag{B5}$$

From Eq.(2.7) of the third paper of Ref.[17] with we get the following expression for the term $\Gamma_a^{(1)}$ describing the HPM gravitational waves

$$\begin{aligned}
\Gamma_a^{(1)}(\tau, \vec{\sigma}) \stackrel{\circ}{=} & -\frac{2G}{c^2} \sum_b \tilde{M}_{ab}^{-1}(\vec{\sigma}) \sum_{uv} \mathcal{P}_{bbuv}(\vec{\sigma}) \sum_i \eta_i \frac{\kappa_{iu}(\tau) \kappa_{iv}(\tau)}{c \sqrt{m_i^2 c^2 + \vec{\kappa}_i^2}} \\
& \left[\frac{1}{|\vec{\sigma} - \vec{\eta}_i(\tau)|} + \sum_{m=1}^{\infty} \frac{1}{(2m)!} \left(\frac{\vec{\kappa}_i(\tau)}{\sqrt{m_i^2 c^2 + \vec{\kappa}_i^2}} \cdot \frac{\partial}{\partial \vec{\sigma}} \right)^{2m} |\vec{\sigma} - \vec{\eta}_i(\tau)|^{2m-1} \right].
\end{aligned} \tag{B6}$$

At the lowest order they reduce to the special relativistic internal Poincaré generators in the rest-frame instant form of Ref.[4]. They are $p_{(1)}^0 = M_{(1)}c = \sum_i \eta_i \sqrt{m_i^2 c^2 + \vec{\kappa}_i^2(\tau)}$, $p_{(1)}^r = \sum_i \eta_i \kappa_{ir}(\tau) \approx 0$, $j_{(1)}^{rs} = \sum_i \eta_i \left(\eta_i^r(\tau) \kappa_{is}(\tau) - \eta_i^s(\tau) \kappa_{ir}(\tau) \right)$, $j_{(1)}^{\tau r} = \sum_i \eta_i \eta_i^r(\tau) \sqrt{m_i^2 c^2 + \vec{\kappa}_i^2(\tau)} \approx 0$. The conditions $j_{(1)}^{\tau r} \approx 0$ and $p_{(1)}^r \approx 0$ are the rest-frame conditions eliminating the 3-center of mass and its conjugate 3-momentum inside the 3-spaces of the rest frame. Like in special relativity there is a decoupled external (canonical but not covariant) 4-center of mass.

The non-relativistic limit of the Poincaré generators are the Galilei generators (B1) restricted to the rest frame and with the Newton center of mass decoupled.

Appendix C: The Non-Relativistic Micro-Canonical Partition Function for N Free Particles

In this Appendix we evaluate the standard non-relativistic micro-canonical partition function for N free particles [see also Ref.[61]] in Subsection 1. Then in the other Subsections we evaluate the new micro-canonical partition function for N free particles in the inertial rest frame defined in Subsection B of Section V. Finally in Subsection 5 we consider the ideal Boltzmann gas.

1. The Standard Micro-Canonical Partition Function with only the Energy Restriction

The partition function (5.1) for equal mass particles with coordinates $\vec{\eta}_i(t)$ and canonical momenta $\vec{\kappa}_i(t)$ ($i = 1, \dots, N$) is

$$Z_{(st,nr)}(\mathcal{E}, V, N) = \frac{1}{N!} \int \prod_{i=1}^N \chi(V) d^3\eta_i d^3\kappa_i \delta\left(\sum_{i=1}^N \frac{\vec{\kappa}_i^2}{2m_i} - \mathcal{E}\right), \quad (C1)$$

in which

$$\chi(V) = 0, \quad \eta_i^2 > R^2, \quad \chi(V) = 1, \quad \eta_i^2 \leq R^2. \quad (C2)$$

We follow the method of Ref.[30] and take the Laplace transform

$$\tilde{Z}_{(nr,st)}(s, V, N) = \int_0^\infty d\mathcal{E} \exp(-s\mathcal{E}) Z(\mathcal{E}, V, N), \quad (C3)$$

so that we get

$$\begin{aligned} \tilde{Z}_{(nr,st)}(s, V, N) &= \frac{1}{N!} \int \prod_{i=1}^N d^3\eta_i d^3\kappa_i \exp\left[-s\left(\sum_{i=1}^N \frac{\vec{\kappa}_i^2}{2m_i}\right)\right] = \\ &= \frac{1}{N!} \int \prod_{i=1}^N d^3\eta_i \left(\sqrt{\frac{2m\pi}{s}}\right)^{3N}. \end{aligned} \quad (C4)$$

The inverse Laplace transform of the function $s^{-n/2}$ is $\frac{(\mathcal{E})^{n/2-1}}{\Gamma(n/2)}\theta(\mathcal{E})$.

Thus with $n = 3N$ we have

$$\begin{aligned} Z_{(nr,st)}(\mathcal{E}, V, N) &= \frac{1}{N!} \frac{(\sqrt{2\pi m})^{3N} \mathcal{E}^{3N/2}}{\mathcal{E} \Gamma(3N/2)} \theta(\mathcal{E}) \int \prod_{i=1}^N d^3\eta_i = \\ &= \frac{1}{N!} \frac{(\sqrt{2\pi m})^{3N} \mathcal{E}^{3N/2}}{\mathcal{E} \Gamma(3N/2)} \theta(\mathcal{E}) V^N. \end{aligned} \quad (C5)$$

This is the standard micro-canonical partition function for N free particles as shown in Ref.[61]

2. Towards the Rest-Frame Micro-Canonical Partition Function: with Energy and Momentum Restrictions Only

To arrive at the partition function $Z_{(nr)}(\mathcal{E}, V, N)$ of Eq.(5.5) we start by evaluating the following function (at the end one will put $\vec{\kappa}_+ = 0$)

$$Z(\mathcal{E}, V, N, \vec{\kappa}_+) = \frac{1}{N!} \int \prod_{i=1}^N d^3\eta_i d^3\kappa_i \delta\left(\sum_{i=1}^N \frac{\vec{\kappa}_i^2}{2m_i} - \mathcal{E}\right) \delta^3\left(\sum_{i=1}^N \vec{\kappa}_i - \vec{\kappa}_+\right). \quad (\text{C6})$$

As before we take the Laplace transform

$$\tilde{Z}(s, V, N, \vec{\kappa}_+) = \int_0^\infty d\mathcal{E} \exp(-s\mathcal{E}) Z(\mathcal{E}, V, N, \vec{\kappa}_+), \quad (\text{C7})$$

and furthermore replace the delta functions by plane wave integrals

$$\tilde{Z}(s, V, N, \vec{\kappa}_+) = \frac{1}{N!(2\pi)^{3N}} \int d^3\lambda \prod_{i=1}^N d^3\eta_i d^3\kappa_i \exp\left[-s\left(\sum_{i=1}^N \frac{\vec{\kappa}_i^2}{2m_i}\right)\right] \times \exp\left[i\vec{\lambda} \cdot \left(\sum_{i=1}^N \vec{\kappa}_i - \vec{\kappa}_+\right)\right]. \quad (\text{C8})$$

Consider the momentum space integral

$$\prod_{i=1}^N \int d^3\kappa_i \exp\left[-s\frac{\vec{\kappa}_i^2}{2m_i} + i\vec{\kappa}_i \cdot \vec{\lambda}\right] = \left(\sqrt{\frac{8\pi^3 m_i^3}{s^3}}\right)^N \exp\left[-\frac{Nm\vec{\lambda}^2}{2s}\right]. \quad (\text{C9})$$

Next we perform the λ integral and we obtain (for equal masses)

$$\tilde{Z}(s, V, N, \vec{\kappa}_+) = \frac{1}{N!(2\pi)^{3N}} \left(\sqrt{\frac{8\pi^3 m^3}{s^3}}\right)^N \sqrt{\frac{8\pi^3 s^3}{N^3 m^3}} \exp\left(-\frac{2s\vec{\kappa}_+^2}{Nm}\right) \prod_{i=1}^N d^3\eta_i. \quad (\text{C10})$$

The inverse Laplace transform of the function $s^{-n/2} \exp(-\frac{s}{s_0})$ is

$$\frac{(\mathcal{E} - 1/s_0)^{n/2-1}}{\Gamma(n/2)} \theta(\mathcal{E} - 1/s_0). \quad (\text{C11})$$

Thus with

$$s_0 = \frac{Nm}{2\vec{\kappa}_+^2}, \quad n = 3N - 3, \quad (\text{C12})$$

we obtain

$$Z(\mathcal{E}, V, N, \vec{\kappa}_+) = \frac{1}{N!} \left(\sqrt{\frac{m}{2\pi}}\right)^{3N} \frac{(\mathcal{E} - 2\vec{\kappa}_+^2/Nm)^{(3N-5)/2}}{\Gamma((3N-3)/2)} \theta(\mathcal{E} - 2\vec{\kappa}_+^2/Nm) \sqrt{\frac{8\pi^3}{N^3 m^3}} \int \prod_{i=1}^N d^3\eta_i. \quad (\text{C13})$$

As with the case above, performing the spatial integrals gives

$$Z(\mathcal{E}, V, N, \vec{\kappa}_+) = \frac{1}{N!} \left(\sqrt{\frac{m}{2\pi}} \right)^{3N-3} \frac{(\mathcal{E} - 2\vec{\kappa}_+^2/Nm)^{(3N-5)/2}}{N^{3/2}\Gamma((3N-3)/2)} \theta(\mathcal{E} - 2\vec{\kappa}_+^2/m) V^N. \quad (\text{C14})$$

3. The Rest-Frame Micro-Canonical Partition Function with Energy, Momentum and Boost Restrictions

The inclusion of the boost restriction modifies Eq.(C6) to the function

$$Z(\mathcal{E}, V, N, \vec{\kappa}_+) = \frac{1}{N!} \int \prod_{i=1}^N d^3\eta_i d^3\kappa_i \delta\left(\sum_{i=1}^N \frac{\vec{\kappa}_i^2}{2m_i} - \mathcal{E}\right) \delta^3\left(\sum_{i=1}^N \vec{\kappa}_i - \vec{\kappa}_+\right) \delta^3\left(\sum_{i=1}^N \vec{\eta}_i m_i\right), \quad (\text{C15})$$

where we have used the non-relativistic form of the boost give in Eq. (2.14) (essentially a center of mass restriction). All is as in the above case leading to the last step we obtain

$$\begin{aligned} Z(\mathcal{E}, V, N, \vec{\kappa}_+) &= \frac{1}{N!} \left(\sqrt{\frac{m}{2\pi}} \right)^{3N} \frac{(\mathcal{E} - 2\vec{\kappa}_+^2/Nm)^{(3N-5)/2}}{\Gamma((3N-3)/2)} \theta(\mathcal{E} - 2\vec{\kappa}_+^2/m) \\ &\quad \sqrt{\frac{8\pi^3}{N^3 m^3}} \int \prod_{i=1}^N \delta^3\left(\sum_{i=1}^N \vec{\eta}_i m_i\right) d^3\eta_i. \\ &= \frac{1}{N!} \left(\sqrt{\frac{m}{2\pi}} \right)^{3N} \frac{(\mathcal{E} - 2\vec{\kappa}_+^2/Nm)^{(3N-5)/2}}{\Gamma((3N-3)/2)} \theta(\mathcal{E} - 2\vec{\kappa}_+^2/m) \\ &\quad \sqrt{\frac{8\pi^3}{N^3 m^3}} \frac{1}{(2\pi)^3} \int \int d^3\rho \prod_{i=1}^N d^3\eta_i \exp(i\vec{\rho} \cdot \sum_{i=1}^N \vec{\eta}_i m). \end{aligned} \quad (\text{C16})$$

For the finite volumes we have spatial integrals of the form $\int d^3\eta \exp(i\vec{\rho} \cdot \vec{\eta}m) = \frac{4\pi}{(\rho m)^3} (\sin R\rho m - R\rho m \cos R\rho m)$. Thus we get $(x = R\rho m)$

$$\begin{aligned}
Z(\mathcal{E}, V, N, \vec{\kappa}_+) &= \\
&= \frac{1}{N!} \left(\sqrt{\frac{m}{2\pi}} \right)^{3N} \frac{(\mathcal{E} - 2\vec{\kappa}_+^2/Nm)^{(3N-5)/2}}{\Gamma((3N-3)/2)} \theta(\mathcal{E} - 2\vec{\kappa}_+^2/m) \\
&\quad \sqrt{\frac{8\pi^3}{N^3 m^3}} \frac{1}{(2\pi)^3} \left(\frac{4\pi}{m^3} \right)^N \int d^3\rho \left(\frac{\sin R\rho m - R\rho m \cos R\rho m}{\rho^3} \right)^N = \\
&= \frac{1}{N!} \left(\sqrt{\frac{m}{2\pi}} \right)^{3N} \frac{(\mathcal{E} - 2\vec{\kappa}_+^2/Nm)^{(3N-5)/2}}{\Gamma((3N-3)/2)} \theta(\mathcal{E} - 2\vec{\kappa}_+^2/m) \\
&\quad \sqrt{\frac{32\pi}{N^3 m^9}} 3^{N-1} V^{N-1} \int_0^\infty x^2 dx \left(\frac{\sin x - x \cos x}{x^3} \right)^N = \\
&= \frac{1}{N!} \left(\sqrt{\frac{m}{2\pi}} \right)^{3N} \frac{(\mathcal{E} - 2\vec{\kappa}_+^2/Nm)^{(3N-5)/2}}{\Gamma((3N-3)/2)} \theta(\mathcal{E} - 2\vec{\kappa}_+^2/Nm) \\
&\quad \sqrt{\frac{32\pi}{N^3 m^9}} 3^{N-1} V^{N-1} \int_0^\infty x^2 dx \left(\frac{j_1(x)}{x} \right)^N. \tag{C17}
\end{aligned}$$

4. The Rest-Frame Extended Micro-Canonical Partition Function with Energy, Momentum, Boost and Angular Momentum Restrictions

The partition function $\tilde{Z}_{(nr)}(\mathcal{E}, V, N, \vec{\mathcal{S}})$ of Eq.(5.5) is obtained by putting $\vec{\kappa}_+ = 0$ in the following function

$$\begin{aligned}
Z(\mathcal{E}, V, N, \vec{\kappa}_+, \vec{\mathcal{S}}) &= \frac{1}{N!} \int \prod_{i=1}^N d^3\eta_i d^3\kappa_i \delta\left(\sum_{i=1}^N \frac{\vec{\kappa}_i^2}{2m_i} - \mathcal{E}\right) \delta^3\left(\sum_{i=1}^N \vec{\eta}_i \times \vec{\kappa}_i - \vec{\mathcal{S}}\right) \\
&\quad \times \delta^3\left(\sum_{i=1}^N \vec{\kappa}_i - \vec{\kappa}_+\right) \delta^3\left(\sum_{i=1}^N \vec{\eta}_i m_i\right). \tag{C18}
\end{aligned}$$

Proceeding as before

$$\tilde{Z}(s, V, N, \vec{\mathcal{S}}) = \int_0^\infty d\mathcal{E} \exp(-s\mathcal{E}) Z(\mathcal{E}, V, N, \vec{\mathcal{S}}), \tag{C19}$$

and

$$\begin{aligned}
&\tilde{Z}(s, V, N, \vec{\kappa}_+, \vec{\mathcal{S}}) = \\
&= \frac{1}{N!(2\pi)^{9N}} \int \int \int d^3\omega d^3\lambda d^3\rho \prod_{i=1}^N d^3\eta_i d^3\kappa_i \exp\left[-s\left(\sum_{i=1}^N \frac{\vec{\kappa}_i^2}{2m_i}\right)\right] \\
&\quad \times \exp\left[i\vec{\omega} \cdot \left(\sum_{i=1}^N \vec{\eta}_i \times \vec{\kappa}_i - \vec{\mathcal{S}}\right)\right] \exp\left[i\vec{\lambda} \cdot \left(\sum_{i=1}^N \vec{\kappa}_i - \vec{\kappa}_+\right)\right] \exp\left(i\vec{\rho} \cdot \sum_{i=1}^N \vec{\eta}_i m_i\right), \tag{C20}
\end{aligned}$$

and

$$\begin{aligned}
& \prod_{i=1}^N \int d^3 \kappa_i \exp[-s \frac{\vec{\kappa}_i^2}{2m_i} + i \vec{\kappa}_i \cdot (\vec{\omega} \times \vec{\eta}_i + \vec{\lambda})] = \\
& = \left(\sqrt{\frac{8\pi^3 m_i^3}{s^3}} \right)^N \exp[-\frac{m \sum_{i=1}^N (\vec{\omega} \times \vec{\eta}_i + \vec{\lambda})^2}{2s}].
\end{aligned} \tag{C21}$$

Thus, assuming equal masses, we get

$$\begin{aligned}
& \tilde{Z}(s, V, N, \vec{\kappa}_+, \vec{\mathcal{S}}) = \\
& = \frac{1}{N!(2\pi)^{9N}} \left(\sqrt{\frac{8\pi^3 m^3}{s^3}} \right)^N \int \int \int d^3 \omega d^3 \lambda d^3 \rho \prod_{i=1}^N d^3 \eta_i \exp[-\frac{m(\vec{\omega} \times \vec{\eta}_i + \vec{\lambda})^2}{2s}] \\
& \times \exp[-i \vec{\omega} \cdot \vec{\mathcal{S}}] \exp[-i \vec{\lambda} \cdot \vec{\kappa}_+] \exp(i \vec{\rho} \cdot \sum_{i=1}^N \vec{\eta}_i m).
\end{aligned} \tag{C22}$$

Next we do the $\vec{\lambda}$ integral

$$\begin{aligned}
& \int d^3 \lambda \prod_{i=1}^N \exp[-\frac{m(\vec{\omega} \times \vec{\eta}_i + \vec{\lambda})^2}{2s}] \exp[-i \vec{\lambda} \cdot \vec{\kappa}_+] = \\
& = \int d^3 \lambda \exp[-\frac{m(\vec{\omega} \times \sum_i \vec{\eta}_i + \vec{\lambda})^2}{2s}] \exp[-i \vec{\lambda} \cdot \vec{\kappa}_+] = \\
& = \int d^3 \lambda' \exp[-\frac{m \vec{\lambda}'^2}{2s}] \exp[-i(\vec{\lambda}' - \vec{\omega} \times \sum_i \vec{\eta}_i) \cdot \vec{\kappa}_+] = \\
& = \exp[i(\vec{\omega} \times \sum_i \vec{\eta}_i) \cdot \vec{\kappa}_+] \int d^3 \lambda' \exp[-\frac{m \vec{\lambda}'^2}{2s}] \exp[-i \vec{\lambda}' \cdot \vec{\kappa}_+],
\end{aligned} \tag{C23}$$

and so we have

$$\begin{aligned}
\tilde{Z}(s, V, N, \vec{\kappa}_+, \vec{\mathcal{S}}) & = \frac{\left(\sqrt{\frac{8\pi^3 m^3}{s^3}} \right)^N \sqrt{\frac{8s^3 \pi^3}{m^3}} \exp[-\frac{2s \vec{\kappa}_+^2}{m}]}{N!(2\pi)^{9N}} \int \int d^3 \omega d^3 \rho \prod_{i=1}^N d^3 \eta_i \\
& \times \exp[i(\vec{\omega} \times \sum_i \vec{\eta}_i) \cdot \vec{\kappa}_+] \exp[-i \vec{\omega} \cdot \vec{\mathcal{S}}] \exp(i \vec{\rho} \cdot \sum_{i=1}^N \vec{\eta}_i m).
\end{aligned} \tag{C24}$$

Consider the $\vec{\eta}$ integrals

$$\begin{aligned}
& \int \prod_{i=1}^N d^3 \eta_i \exp \left[- \sum_{i=1}^N \frac{m(\vec{\omega} \times \vec{\eta}_i)^2}{2s} + i \sum_{i=1}^N \vec{\rho} \cdot \vec{\eta}_i m \right] = \\
& = \int \prod_{i=1}^N d^3 \eta_i \exp \left[-m \sum_{i=1}^N \frac{(\vec{\omega} \times \vec{\eta}_i)^2}{2s} + i \sum_{i=1}^N \vec{\rho} \cdot \vec{\eta}_i m \right] = \\
& = \int \prod_{i=1}^N d^3 \eta_i \exp \left[-m \sum_{i=1}^N \frac{\vec{\omega}^2 \vec{\eta}_i^2 - (\vec{\omega} \cdot \vec{\eta})^2}{2s} + i \sum_{i=1}^N \vec{\rho} \cdot \vec{\eta}_i m \right]. \tag{C25}
\end{aligned}$$

To perform the $\vec{\eta}_i$ integrals we would need to find the transformation that diagonalizes the matrix $\delta_{\alpha\beta} \vec{\omega}^2 - \omega_\alpha \omega_\beta$. The equation

$$\tilde{R}_{\alpha\gamma} (\delta_{\gamma\beta} \vec{\omega}^2 - \omega_\gamma \omega_\beta) R_{\beta\delta} = \omega_\alpha'^2 \delta_{\alpha\delta}, \tag{C26}$$

implies

$$\begin{aligned}
\eta_{ia} &= R_{\alpha\beta} \eta'_{i\beta} = \tilde{R}_{\gamma\alpha} \eta'_{i\gamma}, \\
\vec{\omega}^2 \vec{\eta}_i^2 - (\vec{\omega} \cdot \vec{\eta}_i)^2 &= \eta'_{i\gamma} \tilde{R}_{\gamma\alpha} (\delta_{\alpha\beta} \vec{\omega}^2 - \omega_\alpha \omega_\beta) R_{\beta\delta} \eta'_{i\delta} = \\
&= \eta'_{i\gamma} \omega_\gamma'^2 \delta_{\gamma\delta} \eta'_{i\delta} = \eta'_{i1} \omega_1'^2 \eta'_{i1} + \eta'_{i2} \omega_2'^2 \eta'_{i2} + \eta'_{i3} \omega_3'^2 \eta'_{i3}. \tag{C27}
\end{aligned}$$

Before the rotation, each of the N $\vec{\eta}$ integral is of the form

$$\int_0^R \eta^2 d\eta \int_0^\pi \sin \theta_\eta d\theta_\eta \int_0^{2\pi} d\phi_\eta f(\vec{\eta}). \tag{C28}$$

The volume is spherically symmetric and we have that after the rotation we get ($\det \mathbb{R} = 1$)

$$\begin{aligned}
& \int_0^R \eta'^2 d\eta' \det \mathbb{R} \int_0^\pi \sin \theta'_\eta d\theta'_\eta \int_0^{2\pi} d\phi'_\eta f(\vec{\eta}') = \\
& = \int_0^R \eta'^2 d\eta' \int_0^\pi \sin \theta'_\eta d\theta'_\eta \int_0^{2\pi} d\phi'_\eta f(\vec{\eta}'). \tag{C29}
\end{aligned}$$

And so

$$\begin{aligned}
& \int \prod_{i=1}^N d^3 \eta_i \exp \left[-m \sum_{i=1}^N \frac{\vec{\omega}^2 \vec{\eta}_i^2 - (\vec{\omega} \cdot \vec{\eta})^2 - 2i \vec{\eta}_i \cdot \vec{\rho} s}{2s} \right] = \\
& = \int \prod_{i=1}^N d^3 \eta'_i \exp \left[-m \sum_{i=1}^N \frac{\omega_1'^2 \eta_{i1}'^2 + \omega_2'^2 \eta_{i2}'^2 + \omega_3'^2 \eta_{i3}'^2 - 2i \tilde{\mathbb{R}} \cdot \vec{\eta}'_i \cdot \vec{\rho} s}{2s} \right]. \tag{C30}
\end{aligned}$$

Now

$$\begin{aligned}
-2i \tilde{\mathbb{R}} \cdot \vec{\eta}'_i \cdot \vec{\rho} s &= -2i (\mathcal{R}_{\alpha\beta}) \eta'_{\beta i} (\vec{\rho} s)_\alpha = -2i \eta'_{\beta i} (\vec{\rho} s)_\alpha \mathcal{R}_{\alpha\beta} = \\
&= -2i \vec{\eta}'_{\alpha i} (\vec{\rho}' s)_\alpha = -2i \vec{\eta}'_i \cdot \vec{\rho}', \tag{C31}
\end{aligned}$$

where, since $\det \mathbb{R} = 1$, we have $\vec{\rho}' = \vec{\rho} \cdot \mathbb{R}$.

We now are in a position to set up the $\eta'_{\alpha i}$ integrals. We can drop the primes on the $\vec{\eta}'$ so that we are left with

$$\int \prod_{i=1}^N d^3 \eta_i \exp[-m \sum_{i=1}^N \frac{\omega_1'^2 \eta_{i1}^2 + \omega_2'^2 \eta_{i2}^2 + \omega_3'^2 \eta_{i3}^2}{2s}] \times \exp[im \sum_{i=1}^N \vec{\eta}_i \cdot \vec{\rho}']. \quad (\text{C32})$$

Let us do the $\vec{\omega}$ integral first. It is

$$I = \int d^3 \omega'_i \exp[-m \sum_{i=1}^N \frac{\omega_1'^2 \eta_{i1}^2 + \omega_2'^2 \eta_{i2}^2 + \omega_3'^2 \eta_{i3}^2}{2s}] \exp(-i\vec{\omega}' \cdot \vec{\mathcal{S}}'). \quad (\text{C33})$$

Let us drop the primes and we have

$$\begin{aligned} I &= \int_{-\infty}^{+\infty} d\omega_1 \exp[-\frac{m}{2s} \sum_{i=1}^N (\omega_1^2 \eta_{i1}^2) - i\omega_1 \mathcal{S}'_1] \\ &\times \int_{-\infty}^{+\infty} d\omega_2 \exp[-\frac{m}{2s} \sum_{i=1}^N (\omega_2^2 \eta_{i2}^2) - i\omega_2 \mathcal{S}'_2] \\ &\times \int_{-\infty}^{+\infty} d\omega_3 \exp[-\frac{m}{2s} \sum_{i=1}^N (\omega_3^2 \eta_{i3}^2) - i\omega_3 \mathcal{S}'_3]. \end{aligned} \quad (\text{C34})$$

The first integral is

$$\begin{aligned} &\int_{-\infty}^{+\infty} d\omega_1 \exp[-\frac{m}{2s} \left(\omega_1^2 \sum_{i=1}^N \eta_{i1}^2 \right) - i\omega_1 \mathcal{S}'_1] = \\ &= \exp\left[\left(\frac{m}{2s} \sum_{i=1}^N \eta_{i1}^2\right) a_1^2\right] \sqrt{\frac{2\pi s}{m \sum_{i=1}^N \eta_{i1}^2}}, \\ &a_1 = \frac{is\mathcal{S}'_1}{m \sum_{i=1}^N \eta_{i1}^2}. \end{aligned} \quad (\text{C35})$$

Thus, we now have that

$$\begin{aligned} &\tilde{Z}(s, V, N, \vec{\kappa}_+, \vec{\mathcal{S}}) = \\ &= \frac{1}{N!(2\pi)^{9N}} \left(\sqrt{\frac{8\pi^3 m^3}{s^3}} \right)^N \sqrt{\frac{8s^3 \pi^3}{m^3}} \exp[-\frac{2s\vec{\kappa}_+^2}{m}] \int \int d^3 \rho \prod_{i=1}^N d^3 \eta_i \exp[im \sum_{i=1}^N \vec{\eta}_i \cdot \vec{\rho}] \\ &\times \exp\left[\left(\frac{m}{2s \sum_{i=1}^N \eta_{i1}^2}\right) \left(\frac{is\mathcal{S}'_1}{m}\right)^2\right] \\ &\times \exp\left[\left(\frac{m}{2s \sum_{i=1}^N \eta_{i2}^2}\right) \left(\frac{is\mathcal{S}'_2}{m}\right)^2\right] \\ &\times \exp\left[\left(\frac{m}{2s \sum_{i=1}^N \eta_{i3}^2}\right) \left(\frac{is\mathcal{S}'_3}{m}\right)^2\right]. \end{aligned} \quad (\text{C36})$$

If we perform the ρ integral we get back the boost delta function, so that

$$\begin{aligned}
\tilde{Z}(s, V, N, \vec{\kappa}_+, \vec{\mathcal{S}}) &= \frac{1}{N!(2\pi)^{9N}} \left(\sqrt{\frac{8\pi^3 m^3}{s^3}} \right)^N \sqrt{\frac{8s^3 \pi^3}{m^3}} \exp\left[-\frac{2s\vec{\kappa}_+^2}{m}\right] (2\pi)^3 \int \prod_{i=1}^N d^3 \eta_i \delta^3\left(m \sum_{i=1}^N \vec{\eta}_i\right) \\
&\times \exp\left[\left(\frac{m}{2s \sum_{i=1}^N \eta_{i1}^2}\right) \left(\frac{is\mathcal{S}'_1}{m}\right)^2\right] \\
&\times \exp\left[\left(\frac{m}{2s \sum_{i=1}^N \eta_{i2}^2}\right) \left(\frac{is\mathcal{S}'_2}{m}\right)^2\right] \\
&\times \exp\left[\left(\frac{m}{2s \sum_{i=1}^N \eta_{i3}^2}\right) \left(\frac{is\mathcal{S}'_3}{m}\right)^2\right] = \\
&= \frac{1}{N!(2\pi)^{9N}} \left(\sqrt{\frac{8\pi^3 m^3}{s^3}} \right)^N \sqrt{\frac{8s^3 \pi^3}{m^3}} \exp\left[-\frac{2s\vec{\kappa}_+^2}{m}\right] (2\pi)^3 \int \prod_{i=1}^N d^3 \eta_i \delta^3\left(m \sum_{i=1}^N \vec{\eta}_i\right) \\
&\times \exp\left[-\left(\frac{s\mathcal{S}_1'^2}{2m \sum_{i=1}^N \eta_{i1}^2}\right) - \left(\frac{s\mathcal{S}_2'^2}{2m \sum_{i=1}^N \eta_{i2}^2}\right) - \left(\frac{s\mathcal{S}_3'^2}{2m \sum_{i=1}^N \eta_{i3}^2}\right)\right]. \quad (C37)
\end{aligned}$$

Now we perform the inverse Laplace transform, where the form to be transformed is

$$\exp(-s \tilde{\mathcal{E}}) s^{(3-3N)/2}, \quad \tilde{\mathcal{E}}(\vec{\eta}_i, \vec{\mathcal{S}}) = \sum_{r=1}^3 \frac{(\mathcal{S}^r)^2}{2m \sum_{i=1}^N (\eta_i^r)^2} + \frac{2\vec{\kappa}_+^2}{m}. \quad (C38)$$

Its transform is

$$\frac{(\mathcal{E} - \tilde{\mathcal{E}})^{(3N-3)/2-1}}{\Gamma((3N-3)/2)} \theta(\mathcal{E} - \tilde{\mathcal{E}}), \quad (C39)$$

and consequently

$$\begin{aligned}
Z(\mathcal{E}, V, N, \vec{\kappa}_+, \vec{\mathcal{S}}) &= \\
&= \frac{1}{N!(2\pi)^{9N}} \left(\sqrt{8\pi^3} \right)^{N+1} \left(\sqrt{m^3} \right)^{N-1} (2\pi)^3 \\
&\times \int \prod_{i=1}^N d^3 \eta_i \delta^3\left(m \sum_{i=1}^N \vec{\eta}_i\right) \frac{(\mathcal{E} - \tilde{\mathcal{E}})^{(3N-3)/2-1}}{\Gamma((3N-3)/2)} \theta(\mathcal{E} - \tilde{\mathcal{E}}), \\
\tilde{\mathcal{E}} &= \sum_{r=1}^3 \frac{\mathcal{S}_r^2}{2m \sum_{i=1}^N \eta_{ir}^2} + \frac{2\vec{\kappa}_+^2}{m}. \quad (C40)
\end{aligned}$$

Each integral is of the form

$$\begin{aligned}
\int d^3\eta &= \int_0^R \eta^2 d\eta \int_0^\pi \sin \theta_\eta d\theta_\eta \int_0^{2\pi} d\phi_\eta, \quad \eta = R x, \\
\int d^3\eta &= R^3 \int_0^1 x^2 dx \int_0^\pi \sin \theta_\eta d\theta_\eta \int_0^{2\pi} d\phi_\eta = \\
&= \frac{3V}{4\pi} \int_0^1 x^2 dx \int_0^\pi \sin \theta_\eta d\theta_\eta \int_0^{2\pi} d\phi_\eta.
\end{aligned} \tag{C41}$$

Thus, with

$$\begin{aligned}
\tilde{\mathcal{E}} &= \frac{(\mathcal{S}^1)^2}{2m \sum_{i=1}^N \eta_{i1}^2} + \frac{(\mathcal{S}^2)^2}{2m \sum_{i=1}^N \eta_{i2}^2} + \frac{(\mathcal{S}^3)^2}{2m \sum_{i=1}^N \eta_{i3}^2} + \frac{2\vec{\kappa}_+^2}{m} = \\
&= \frac{1}{2mR^2} \left(\frac{(\mathcal{S}^1)^2}{\sum_{i=1}^N x_i^2 \cos^2 \phi_i \sin^2 \theta_i} + \frac{(\mathcal{S}^2)^2}{\sum_{i=1}^N x_i^2 \sin^2 \phi_i \sin^2 \theta_i} + \frac{(\mathcal{S}^3)^2}{2m \sum_{i=1}^N x_i^2 \cos^2 \theta_i} \right) + \frac{2\vec{\kappa}_+^2}{m},
\end{aligned} \tag{C42}$$

we have

$$\begin{aligned}
Z(\mathcal{E}, V, N, \vec{\kappa}_+, \vec{\mathcal{S}}) &= \\
&= \frac{1}{N!(2\pi)^{9N}} \left(\sqrt{8\pi^3} \right)^{N+1} \left(\sqrt{m^3} \right)^{N-1} \frac{(2\pi)^3}{(m)^3} \left(\frac{3V}{4\pi} \right)^{N-1} \\
&\times \prod_i^N \int_0^1 x_i^2 dx_i \int_0^\pi \sin \theta_i d\theta_i \int_0^{2\pi} d\phi_i \delta^3 \left(\sum_{i=1}^N \vec{x}_i \right) \\
&\frac{(\mathcal{E} - \tilde{\mathcal{E}}(\vec{\eta}_i, \vec{\mathcal{S}}))^{(3N-3)/2-1}}{\Gamma((3N-3)/2)} \theta(\mathcal{E} - \tilde{\mathcal{E}}(\vec{\eta}_i, \vec{\mathcal{S}})).
\end{aligned} \tag{C43}$$

5. The Ideal Boltzmann Gas

In this subsection we take the partition functions given in the above subsections and see the connections between temperature and internal energy and the ideal gas law. For the standard micro-canonical partition function with only the energy restriction, Eq. (C5) with $S_{(nr,st)}(\mathcal{E}, V, N) = k_B \ln Z_{(nr,st)}(\mathcal{E}, V, N)$ leads to

$$\begin{aligned}
\frac{1}{k_B T} &= \left[\frac{\partial S_{(nr,st)}(\mathcal{E}, V, N)}{\partial \mathcal{E}} \right]_{V,N} = k_B \left[\frac{\partial Z_{(nr,st)}(\mathcal{E}, V, N) / \partial \mathcal{E}}{Z_{(nr,st)}(\mathcal{E}, V, N)} \right]_{V,N} = k_B \frac{3N/2 - 1}{\mathcal{E}} \rightarrow k_B \frac{3N}{2\mathcal{E}}, \\
\frac{P}{T} &= \left[\frac{\partial S_{(nr,st)}(\mathcal{E}, V, N)}{\partial V} \right]_{\mathcal{E},N} = k_B \left[\frac{\partial Z_{(nr,st)}(\mathcal{E}, V, N) / \partial V}{Z_{(nr,st)}(\mathcal{E}, V, N)} \right]_{\mathcal{E},N} = k_B \frac{N}{V}.
\end{aligned} \tag{C44}$$

From Eq. (C14) with $\vec{\kappa}_+ = 0$ we obtain with energy and momentum restrictions only in the rest frame

$$\begin{aligned}
\frac{1}{k_B T} &= \left[\frac{\partial S_{(nr,st)}(\mathcal{E}, V, N, \vec{0})}{\partial \mathcal{E}} \right]_{V,N} = k_B \left[\frac{\partial Z_{(nr,st)}(\mathcal{E}, V, N, \vec{0}) / \partial \mathcal{E}}{Z_{(nr,st)}(\mathcal{E}, V, N, \vec{0})} \right]_{V,N} = k_B \frac{3N/2 - 5/2}{\mathcal{E}} \rightarrow k_B \frac{3N}{2\mathcal{E}}, \\
\frac{P}{T} &= \left[\frac{\partial S_{(nr,st)}(\mathcal{E}, V, N, \vec{0})}{\partial V} \right]_{\mathcal{E},N} = k_B \left[\frac{\partial Z_{(nr,st)}(\mathcal{E}, V, N, \vec{0}) / \partial V}{Z_{(nr,st)}(\mathcal{E}, V, N, \vec{0})} \right]_{\mathcal{E},N} = k_B \frac{N}{V}. \quad (C45)
\end{aligned}$$

From Eq. (C16) with $\vec{\kappa}_+ = 0$ we obtain with energy, momentum, and boost restrictions only in the rest frame

$$\begin{aligned}
\frac{1}{k_B T} &= \left[\frac{\partial S_{(nr,st)}(\mathcal{E}, V, N, \vec{0})}{\partial \mathcal{E}} \right]_{V,N} = k_B \left[\frac{\partial Z_{(nr,st)}(\mathcal{E}, V, N, \vec{0}) / \partial \mathcal{E}}{Z_{(nr,st)}(\mathcal{E}, V, N, \vec{0})} \right]_{V,N} = k_B \frac{3N/2 - 5/2}{\mathcal{E}} \rightarrow k_B \frac{3N}{2\mathcal{E}}, \\
\frac{P}{T} &= \left[\frac{\partial S_{(nr,st)}(\mathcal{E}, V, N, \vec{0})}{\partial V} \right]_{\mathcal{E},N} = k_B \left[\frac{\partial Z_{(nr,st)}(\mathcal{E}, V, N, \vec{0}) / \partial V}{Z_{(nr,st)}(\mathcal{E}, V, N, \vec{0})} \right]_{\mathcal{E},N} = k_B \frac{N-1}{V} \rightarrow k_B \frac{N}{V}. \quad (C46)
\end{aligned}$$

From Eq. (C43) we obtain with energy, momentum, boost, and angular momentum restrictions only in the rest frame

$$\begin{aligned}
\frac{1}{k_B T} &= \left[\frac{\partial S_{(nr,st)}(\mathcal{E}, V, N, \vec{0}, \vec{S})}{\partial \mathcal{E}} \right]_{V,N} = k_B \left[\frac{\partial Z_{(nr,st)}(\mathcal{E}, V, N, \vec{0}, \vec{S}) / \partial \mathcal{E}}{Z_{(nr,st)}(\mathcal{E}, V, N, \vec{0}, \vec{S})} \right]_{V,N} = k_B \frac{3N/2 - 5/2}{\mathcal{E}} \rightarrow k_B \frac{3N}{2\mathcal{E}}, \\
\frac{P}{T} &= \left[\frac{\partial S_{(nr,st)}(\mathcal{E}, V, N, \vec{0}, \vec{S})}{\partial V} \right]_{\mathcal{E},N} = k_B \left[\frac{\partial Z_{(nr,st)}(\mathcal{E}, V, N, \vec{0}, \vec{S}) / \partial V}{Z_{(nr,st)}(\mathcal{E}, V, N, \vec{0}, \vec{S})} \right]_{\mathcal{E},N} = k_B \frac{N-1}{V} \rightarrow k_B \frac{N}{V}. \quad (C47)
\end{aligned}$$

All results for large N lead to the ideal gas law forms of

$$\begin{aligned}
\mathcal{E} &= \frac{3}{2} k_B T, \\
PV &= k_B NT. \quad (C48)
\end{aligned}$$

6. The Relativistic Micro-Canonical Partition Function for N Free Particles with only the Energy Restriction

The desired partition function for equal mass particles with coordinates $\vec{\eta}_i(t)$ and canonical momenta $\vec{\kappa}_i(t)$ ($i = 1, \dots, N$) is

$$Z_{(st,nr)}(\mathcal{E}, V, N) = \frac{1}{N!} \int \prod_{i=1}^N \chi(V) d^3 \eta_i d^3 \kappa_i \delta \left(\sum_{i=1}^N \sqrt{\vec{\kappa}_i^2 c^2 + m_i^2 c^4} - \mathcal{E} \right), \quad (C49)$$

in which

$$\chi(V) = 0, \quad \eta_i^2 > R^2, \quad \chi(V) = 1, \quad \eta_i^2 \leq R^2. \quad (C50)$$

We follow the method of [30] and take the Laplace transform

$$\tilde{Z}_{(nr,st)}(s, V, N) = \int_0^\infty d\mathcal{E} \exp(-s\mathcal{E}) Z(\mathcal{E}, V, N), \quad (\text{C51})$$

so that we get

$$\begin{aligned} \tilde{Z}_{(nr,st)}(s, V, N) &= \frac{1}{N!} \int \prod_{i=1}^N d^3\eta_i d^3\kappa_i \exp[-s(\sum_{i=1}^N \sqrt{\vec{\kappa}_i^2 c^2 + m_i^2 c^4})] = \\ &= \frac{1}{N!} \int \prod_{i=1}^N d^3\eta_i (4\pi)^N \int_0^\infty d\kappa_i \kappa_i^2 \exp[-s(\sqrt{\kappa_i^2 c^2 + m_i^2 c^4})] \end{aligned} \quad (\text{C52})$$

From integral tables we get

$$\int_0^\infty d\kappa_i \kappa_i^2 \exp[-s(\sqrt{\kappa_i^2 c^2 + m_i^2 c^4})] = \frac{m_i^2 c}{s} K_2(sm_i c^2). \quad (\text{C53})$$

Thus, with equal masses we get

$$\begin{aligned} \tilde{Z}_{(nr,st)}(s, V, N) &= \frac{1}{N!} \int \prod_{i=1}^N d^3\eta_i d^3\kappa_i \exp[-s(\sum_{i=1}^N \sqrt{\vec{\kappa}_i^2 c^2 + m_i^2 c^4})] = \\ &= \frac{1}{N!} \int \prod_{i=1}^N d^3\eta_i (4\pi)^N \left(\frac{m^2 c}{s} K_2(sm c^2) \right)^N \\ &\quad \frac{(4\pi V)^N}{N!} \left(\frac{m^2 c}{s} K_2(sm c^2) \right)^N. \end{aligned} \quad (\text{C54})$$

Unfortunately there does not appear an analytic expression for the inverse Laplace transform of the N th power of $\frac{m^2 c}{s} K_2(sm c^2)$.

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